Sheet 5 – Solutions

MA180 / MA185 / MA190 Mathematics

C1. (a) For $x \neq 0$, the product rule gives

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3}(x-3) + x^{2/3} \\ &= \frac{2(x-3)+3x}{3x^{1/3}} = \frac{5x-6}{3x^{1/3}}. \end{aligned}$$

(b) The derivative vanishes when $5x - 6 = 0$, giving $x = 6/5$. It is undefined at $x = 0$, although $f(0) = 0$ is defined. Indeed, the difference quotient

$$\frac{f(h) - f(0)}{h} = \frac{h-3}{h^{1/3}}$$

has no finite limit as $h \to 0$. Thus the critical points are $x = 0$ and $x = 6/5$.

(c) For $x < 0$, both $5x-6$ and $x^{1/3}$ are negative, so $f'(x) > 0$. For $0 < x < 6/5$ we have $f'(x) < 0$, and for $x > 6/5$ we have $f'(x) > 0$. Hence f increases, then decreases, then increases. Therefore $x = 0$ is a local maximum, with $f(0) = 0$, and $x = 6/5$ is a local minimum, with

$$f\left(\frac{6}{5}\right) = -\frac{9}{5}\left(\frac{6}{5}\right)^{2/3}.$$

C2. (a) Differentiating,

$$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3).$$

Thus the critical points in $[-2, 5]$ are $x = -1$ and $x = 3$.

(b) At the critical points and endpoints,

$$\begin{aligned} f(-2) = 0, \qquad f(-1) = 7, \\ f(3) = -25, \qquad f(5) = 7. \end{aligned}$$

(c) The absolute maximum value is 7, attained at both $x = -1$ and $x = 5$. The absolute minimum value is -25, attained at $x = 3$. Since f is continuous on the closed interval $[-2, 5]$, the Extreme Value Theorem guarantees both extrema. They must occur among the critical points and endpoints, all of which have been checked.

C3. (a) The surface-area constraint gives

$$h = \frac{6 - 2x^2}{4x} = \frac{3 - x^2}{2x}.$$

Since $x > 0$ and $h > 0$, we require $0 < x < \sqrt{3}$. Thus

$$V(x) = x^2\left(\frac{3-x^2}{2x}\right) = \frac{3x - x^3}{2}, \qquad 0 < x < \sqrt{3}.$$

(b) Differentiating,

$$V'(x) = \frac{3 - 3x^2}{2}.$$

Hence $V'(x) = 0$ when $x^2 = 1$. As $x > 0$, the stationary point is $x = 1$.

(c) For $0 < x < 1$, $V'(x) > 0$, while for $1 < x < \sqrt{3}$, $V'(x) < 0$. Hence the volume increases up to $x = 1$ and decreases thereafter, so this gives the maximum possible volume. At $x = 1$,

$$h = \frac{3 - 1}{2} = 1, \qquad V(1) = 1.$$

The box has dimensions $1\,\mathrm{m} \times 1\,\mathrm{m} \times 1\,\mathrm{m}$, with maximum volume $1\,\mathrm{m}^3$.

A1. (a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ 1 & 3 & 2 & 4 \\ 3 & 2 & -1 & 1 \end{array}\right).$$

(b) In $A\mathbf{v} = \mathbf{b}$, we have

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}.$$

(c) The elementary row operations are: interchange two rows; multiply a row by a non-zero constant; and add a multiple of one row to another row.

A2. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & 9 \\ -1 & 2 & 1 & -1 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 5 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 5 \\ 0 & 0 & 3 & 6 \end{array}\right).$$

Back-substitution gives $3z = 6$, so $z = 2$. The second row gives $-3y + 2 = 5$, so $y = -1$. Finally, $x + y + z = 2$ gives $x = 1$. Thus $(x, y, z) = (1, -1, 2)$.

A3. (a) Row reduction gives

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 10 \\ 0 & 1 & 3 & 4 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The final matrix is in RREF.

(b) The equations are

$$x - 7z = -5, \qquad y + 3z = 4, \qquad 0 = 0.$$

Taking $z = t$, where $t \in \mathbb{R}$, gives all solutions:

$$x = -5 + 7t, \qquad y = 4 - 3t, \qquad z = t.$$

(c) The last row gives no restriction, and z is a free variable. Every $t \in \mathbb{R}$ gives a solution, so there are infinitely many solutions.

A4. (a) The rows give $x = -1$, $y = 3$. There is a unique solution $(-1, 3)$.

(b) The second row says $0 = 1$, which is impossible. Thus there is no solution.

(c) The only non-trivial equation is $x - 3y = 4$. Taking $y = t \in \mathbb{R}$ gives

$$x = 4 + 3t, \qquad y = t.$$

Hence there are infinitely many solutions.

A5. (a) Substituting the three points gives

$$\begin{cases} c = 1, \\ a + b + c = 4, \\ 4a + 2b + c = 11. \end{cases}$$

(b) Since $c = 1$, the other equations become $a + b = 3$ and $2a + b = 5$. Subtraction gives $a = 2$, then $b = 1$. Therefore the parabola is

$$y = 2x^2 + x + 1.$$

(c) At $x = 0, 1, 2$ this expression gives, respectively, 1, $2+1+1 = 4$ and $8+2+1 = 11$. Thus all three points lie on the parabola.

A6. (a) The dimensions are 2×3 for A and 3×2 for B.

(b) Reading the specified rows and columns,

$$A_{12} = -2, \quad A_{23} = 5, \quad B_{21} = -1, \quad B_{32} = 0.$$

(c) The sum $A + B$ is undefined because the matrices have different dimensions. The product AB is defined and has dimensions 2×2, while BA is defined and has dimensions 3×3.

A7. The matrices all have dimensions 2×3, so the operations are defined entry by entry:

$$A + B = \begin{pmatrix} 3 & 1 & 5 \\ -3 & 5 & 2 \end{pmatrix},$$

$$2A - B = \begin{pmatrix} 0 & 5 & -5 \\ -6 & -2 & 10 \end{pmatrix},$$

$$-3B = \begin{pmatrix} -6 & 3 & -15 \\ 0 & -12 & 6 \end{pmatrix}.$$

Each resulting matrix is 2×3.

A8. (a) Multiplying rows by columns,

$$AB = \begin{pmatrix} 1(2) + 2(3) & 1(0) + 2(1) \\ 0(2) + 1(3) & 0(0) + 1(1) \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 3 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 2(1) + 0(0) & 2(2) + 0(1) \\ 3(1) + 1(0) & 3(2) + 1(1) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}.$$

(b) Since $AB \neq BA$, matrix multiplication is not commutative in general.

A9. (a) The identity and zero matrices are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Direct calculation gives

$$AI = IA = A + Z = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(b) Multiplying rows by columns,

$$AD = \begin{pmatrix} 2 & -2 \\ 6 & -4 \end{pmatrix}, \qquad DA = \begin{pmatrix} 2 & 4 \\ -3 & -4 \end{pmatrix}.$$

Thus $AD \neq DA$.

(c) The matrix identities are

$$A + Z = A, \qquad AI = A, \qquad AZ = Z.$$

A10. (a) Using $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$,

$$\begin{aligned} \det M &= 2(3) - 1(5) = 1, \\ \det N &= 2(2) - 4(1) = 0, \\ \det P &= 0(1) - 2(-3) = 6. \end{aligned}$$

(b) A 2×2 matrix is invertible exactly when its determinant is non-zero. Thus M and P are invertible, whereas N is not.

A11. (a) We have $\det A = 2(1) - 1(1) = 1 \neq 0$, so A is invertible.

(b) Suppose $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. The equation $AB = I$ gives

$$2p + r = 1, \quad p + r = 0, \quad 2q + s = 0, \quad q + s = 1.$$

Hence $p = 1$, $r = -1$, $q = -1$ and $s = 2$, so

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Indeed,

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = BA.$$

Therefore $A^{-1} = B$.

(c) Multiplication gives

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The check is

$$A\mathbf{x} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \mathbf{b}.$$

Thus the solution is $(x, y) = (3, 1)$.