Sheet 5

MA180 / MA185 / MA190 Mathematics

C1. Consider the function

$$f(x) = x^{2/3}(x-3).$$

(a) Find $f'(x)$ for $x \neq 0$.

(b) Find all critical points of f, remembering that a critical point can occur where the derivative does not exist.

(c) Determine whether each critical point is a local maximum, a local minimum, or neither. Justify your answers.

C2. Let

$$f(x) = x^3 - 3x^2 - 9x + 2, \qquad -2 \leq x \leq 5.$$

(a) Find all critical points of f in the interval $[-2, 5]$.

(b) Evaluate f at the critical points and at both endpoints.

(c) Find the absolute maximum and absolute minimum values of f on $[-2, 5]$, stating where each occurs. Explain why these are the absolute extrema.

C3. A closed box with a square base is to be made using $6\,\mathrm{m}^2$ of cardboard. Let $x > 0$ metres be the side length of its base and $h > 0$ metres its height. The total surface area and volume are

$$2x^2 + 4xh = 6, \qquad V = x^2 h.$$

(a) Express h in terms of x, and hence write V as a function of x alone. State the possible values of x.

(b) Differentiate $V(x)$ and find its stationary point.

(c) Show that this gives the maximum possible volume, and find the dimensions of the box.

A1. Consider the system of linear equations

$$\begin{cases} 2x - y + z = 7, \\ x + 3y + 2z = 4, \\ 3x + 2y - z = 1. \end{cases}$$

(a) Write its augmented matrix.

(b) Write the system in the form $A\mathbf{v} = \mathbf{b}$, identifying A, $\mathbf{v}$ and $\mathbf{b}$.

(c) State the three elementary row operations that can be used to solve a linear system.

A2. Use Gaussian elimination to solve

$$\begin{cases} x + y + z = 2, \\ 2x - y + 3z = 9, \\ -x + 2y + z = -1. \end{cases}$$

Show your row operations and use back-substitution to find x, y and z.

A3. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 10 \\ 0 & 1 & 3 & 4 \end{array}\right).$$

(a) Use row operations to put the matrix in reduced row echelon form (RREF).

(b) Write down the corresponding equations and find all solutions, using a parameter $t \in \mathbb{R}$.

(c) Explain why the system has infinitely many solutions.

A4. For each augmented matrix below, determine whether its system has a unique solution, no solution, or infinitely many solutions. In the cases where solutions exist, describe them.

(a) $\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right)$;

(b) $\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 1 \end{array}\right)$;

(c) $\left(\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 0 & 0 \end{array}\right)$.

A5. A parabola has equation $y = ax^2 + bx + c$ and passes through the points $(0, 1)$, $(1, 4)$ and $(2, 11)$.

(a) Form a system of three linear equations for a, b and c.

(b) Solve the system to find the equation of the parabola.

(c) Check that your equation passes through all three points.

A6. Let

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ -1 & 4 \\ 3 & 0 \end{pmatrix}.$$

(a) Give the dimensions of A and B.

(b) Find the entries A_{12}, A_{23}, B_{21} and B_{32}.

(c) Determine which of $A + B$, AB and BA are defined. State the dimensions of each product that is defined.

A7. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 4 & -2 \end{pmatrix}.$$

Calculate $A + B$, $2A - B$ and $-3B$. State the dimensions of each resulting matrix.

A8. Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}.$$

(a) Calculate AB and BA, showing how you obtain the entries.

(b) Explain what your answers show about the commutativity of matrix multiplication.

A9. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let I be the 2×2 identity matrix and Z the 2×2 zero matrix.

(a) Calculate AI, IA and $A + Z$.

(b) Calculate AD and DA. Are they equal?

(c) State the matrix identities for $A + Z$, AI and AZ.

A10. Recall that the determinant of a 2×2 matrix is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Consider

$$M = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \qquad N = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}.$$

(a) Calculate $\det M$, $\det N$ and $\det P$.

(b) Determine which matrices are invertible, and explain how their determinants give the answer.

A11. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}.$$

(a) Calculate $\det A$ and explain why A^{-1} exists.

(b) Find A^{-1} by finding a matrix B such that $AB = BA = I$.

(c) Use $\mathbf{x} = A^{-1}\mathbf{b}$ to solve $A\mathbf{x} = \mathbf{b}$, and check your answer by multiplication.