Sheet 4 – Solutions

MA180 / MA185 / MA190 Mathematics

C1. (a) With $u = 5x$, we have

$$\frac{\sin(5x)}{x} = 5\frac{\sin u}{u}.$$

As $x \to 0$, also $u \to 0$, so the limit is 5.

(b) With $u = 3x$,

$$\frac{1-\cos(3x)}{x} = -3\frac{\cos u - 1}{u}.$$

The limit is therefore 0.

(c) For $x \neq 0$ sufficiently close to zero,

$$\frac{\sin(4x)}{\sin(7x)} = \frac{4}{7}\,\frac{\sin(4x)}{4x}\,\frac{7x}{\sin(7x)}.$$

Each of the last two factors tends to 1, so the limit is $4/7$.

C2. (a) By the definition of the derivative,

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\left(\cos x\frac{\cos h - 1}{h} - \sin x\frac{\sin h}{h}\right)\\ &= \cos x\cdot 0 - \sin x\cdot 1 = -\sin x.\end{aligned}$$

We used the cosine addition identity and the two standard trigonometric limits.

(b) Wherever $\cos x \neq 0$, the quotient rule gives

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

The derivative is defined for $x \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$.

C3. (a) The compositions are

$$(f\circ g)(x) = f(\sin x) = \sin^2 x + 2,$$

$$(g\circ f)(x) = g(x^2+2) = \sin(x^2+2).$$

(b) The chain rule gives

$$(f\circ g)'(x) = 2\sin x\cos x = \sin(2x),$$

$$(g\circ f)'(x) = 2x\cos(x^2+2).$$

(c) The functions are not the same: for instance, $(f\circ g)(0) = 2$, whereas $(g\circ f)(0) = \sin 2 \neq 2$.

C4. (a) Let $u = x^2 - 3x + 4$. Then $u' = 2x - 3$, so

$$y' = 6u^5u' = 6(2x-3)(x^2-3x+4)^5.$$

(b) Let $u = 5x^3 - 2$. Then $u' = 15x^2$, so

$$y' = -2u^{-3}u' = -\frac{30x^2}{(5x^3-2)^3}.$$

(c) Write $y = (4x^2+1)^{1/2}$. Thus

$$y' = \frac{1}{2}(4x^2+1)^{-1/2}(8x) = \frac{4x}{\sqrt{4x^2+1}}.$$

C5. (a) We have $f(x) = (x^2+4)^{2/3}$. By the chain rule,

$$f'(x) = \frac{2}{3}(x^2+4)^{-1/3}(2x) = \frac{4x}{3\sqrt[3]{x^2+4}}.$$

Since $x^2 + 4 > 0$, the derivative exists for all $x \in \mathbb{R}$.

(b) Write $g(x) = (3x^2+1)^{1/4}$. Then

$$g'(x) = \frac{1}{4}(3x^2+1)^{-3/4}(6x) = \frac{3x}{2(3x^2+1)^{3/4}}.$$

Again, the derivative exists for all $x \in \mathbb{R}$.

(c) Write $h(x) = (2x+5)^{-3/2}$. Then

$$h'(x) = -\frac{3}{2}(2x+5)^{-5/2}(2) = -\frac{3}{(2x+5)^{5/2}}.$$

As a real-valued function, h and its derivative are defined only for $x > -5/2$.

C6. (a) $y' = 4\cos(4x-1)$.

(b) $y' = -6x\sin(3x^2)$.

(c) $y' = (2x+2)\sec^2(x^2+2x)$, wherever $\cos(x^2+2x) \neq 0$.

(d) Since $y = (\sin x)^2$,

$$y' = 2\sin x\cos x = \sin(2x).$$

C7. (a)

$$f'(x) = 2x\sin x + x^2\cos x.$$

(b) The product and chain rules give

$$g'(x) = 2x\cos(3x) - 3(x^2+1)\sin(3x).$$

(c)

$$\begin{aligned} h'(x) &= (\cos x)(\cos x) + (\sin x)(-\sin x) \\ &= \cos^2 x - \sin^2 x = \cos(2x). \end{aligned}$$

C8. (a) By the quotient rule,

$$f'(x) = \frac{(x+1)\cos x - \sin x}{(x+1)^2}.$$

Both f and f' are defined for $x \neq -1$.

(b) The quotient rule gives

$$\begin{aligned} g'(x) &= \frac{(1+\cos x) - x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{1+\cos x + x\sin x}{(1+\cos x)^2}. \end{aligned}$$

We require $1 + \cos x \neq 0$, so the domain is $x \neq (2k+1)\pi$ for $k \in \mathbb{Z}$.

C9. (a) The product and chain rules give

$$y' = 6x^2(x^3+1)\sin x + (x^3+1)^2\cos x.$$

(b)

$$y' = \cos(x^2) - 2x^2\sin(x^2).$$

(c) Rewrite $y = \sin x\,(x^2+1)^{-1/2}$. Then

$$\begin{aligned} y' &= \cos x\,(x^2+1)^{-1/2} - x\sin x\,(x^2+1)^{-3/2} \\ &= \frac{(x^2+1)\cos x - x\sin x}{(x^2+1)^{3/2}}. \end{aligned}$$

C10. (a) The volume condition gives

$$\pi r^2 h = 128\pi, \qquad h = \frac{128}{r^2}.$$

Substituting this into the area formula,

$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right) = 2\pi r^2 + \frac{256\pi}{r}, \qquad r > 0.$$

(b) Differentiate:

$$S'(r) = 4\pi r - \frac{256\pi}{r^2} = \frac{4\pi(r^3-64)}{r^2}.$$

Since $r > 0$, the stationary point satisfies $r^3 = 64$, giving $r = 4$.

(c) The denominator of $S'(r)$ is positive for $r > 0$. Thus $S'(r) < 0$ for $0 < r < 4$, and $S'(r) > 0$ for $r > 4$. The area decreases and then increases, so $r = 4$ gives a minimum. Finally,

$$h = \frac{128}{4^2} = 8.$$

The required dimensions are radius 4 cm and height 8 cm.

A1. (a) Since $84 = 2^2 \cdot 3 \cdot 7$ and $100 = 2^2 \cdot 5^2$,

$$\phi(84) = \phi(2^2)\phi(3)\phi(7) = 2(2-1)(3-1)(7-1) = 24,$$
$$\phi(100) = \phi(2^2)\phi(5^2) = 2(2-1)5(5-1) = 40.$$

(b) As $\gcd(7, 100) = 1$, Euler's theorem gives $7^{40} \equiv 1 \pmod{100}$. Since $123 = 3(40) + 3$,

$$7^{123} \equiv 7^3 = 343 \equiv 43 \pmod{100}.$$

Therefore the last two digits are 43.

A2. (a) Since $\gcd(7, 26) = 1$, the coefficient 7 has an inverse in $\mathbb{Z}_{26}$. Hence the affine map is invertible.

(b) Euclid's algorithm gives

$$26 = 3(7) + 5, \quad 7 = 5 + 2, \quad 5 = 2(2) + 1.$$

Back-substituting,

$$\begin{aligned} 1 &= 5 - 2(2) = 5 - 2(7 - 5) \\ &= 3(5) - 2(7) = 3(26 - 3(7)) - 2(7) \\ &= 3(26) - 11(7). \end{aligned}$$

Thus $7^{-1} \equiv -11 \equiv 15 \pmod{26}$. To decrypt $y \equiv 7x + 3 \pmod{26}$, we obtain

$$x \equiv 15(y - 3) \pmod{26}.$$

Therefore $f_D(y) \equiv 15(y - 3) \pmod{26}$.

(c) The letters of `CODE` correspond to $3, 15, 4, 5$. Encrypting gives

Letter	C	O	D	E
x	3	15	4	5
$7x + 3 \pmod{26}$	24	4	5	12
Cipher letter	X	D	E	L

Thus the ciphertext is `XDEL`. Applying $f_D(y) = 15(y - 3)$ to $24, 4, 5, 12$ gives residues $3, 15, 4, 5$, respectively, and recovers `CODE`.

A3. (a) Since $A = 1$ is encrypted as $H = 8$ and $B = 2$ as $M = 13$, we have

$$a + b \equiv 8 \pmod{26}, \qquad 2a + b \equiv 13 \pmod{26}.$$

Subtracting gives $a \equiv 5 \pmod{26}$, and hence $b \equiv 8 - 5 \equiv 3 \pmod{26}$. Thus

$$f_E(x) \equiv 5x + 3 \pmod{26}.$$

(b) Since $\gcd(5, 26) = 1$, the coefficient 5 has a multiplicative inverse modulo 26. Indeed,

$$26 = 5(5) + 1, \qquad 1 = 26 - 5(5),$$

so $5^{-1} \equiv -5 \equiv 21 \pmod{26}$. Solving $y \equiv 5x + 3 \pmod{26}$ for x gives

$$f_D(y) \equiv 21(y - 3) \pmod{26}.$$

(c) The ciphertext `QBKKZ` has numerical values $17, 2, 11, 11, 26$. Applying the decryption function gives

Cipher letter	Q	B	K	K	Z
y	17	2	11	11	26
$21(y-3) \pmod{26}$	8	5	12	12	15
Plain letter	H	E	L	L	O

Hence the plaintext is `HELLO`. To check, encrypting $8, 5, 12, 12, 15$ using $5x + 3$ gives $17, 2, 11, 11, 26$ modulo 26, recovering `QBKKZ`.