

Sheet 4

MA180 / MA185 / MA190 Mathematics

C1. Use the standard trigonometric limits

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1, \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$$

to evaluate the following limits. Show your working.

(a) $\lim_{x \rightarrow 0} \frac{\sin(5x)}{x};$

(b) $\lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x};$

(c) $\lim_{x \rightarrow 0} \frac{\sin(4x)}{\sin(7x)}.$

C2. (a) Starting from the limit definition of the derivative and using $\cos(x + h) = \cos x \cos h - \sin x \sin h$, prove that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

(b) Using $\tan x = \frac{\sin x}{\cos x}$ and the quotient rule, show that $\frac{d}{dx}(\tan x) = \sec^2 x$ wherever $\tan x$ is defined.

C3. Let $f(x) = x^2 + 2$ and $g(x) = \sin x$.

(a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

(b) Differentiate both composite functions using the chain rule.

(c) Explain briefly why $f \circ g$ and $g \circ f$ are not the same function.

C4. Use the chain rule to differentiate each function.

(a) $y = (x^2 - 3x + 4)^6;$

(b) $y = (5x^3 - 2)^{-2};$

(c) $y = \sqrt{4x^2 + 1}.$

C5. Rewrite the functions using rational powers, then differentiate. State any restrictions on x where the derivative does not exist.

(a) $f(x) = \sqrt[3]{(x^2 + 4)^2};$

(b) $g(x) = \sqrt[4]{3x^2 + 1};$

(c) $h(x) = \frac{1}{(2x + 5)^{3/2}}.$

C6. Differentiate each trigonometric function using the chain rule.

- (a) $y = \sin(4x - 1)$;
- (b) $y = \cos(3x^2)$;
- (c) $y = \tan(x^2 + 2x)$;
- (d) $y = \sin^2 x$.

C7. Use the product rule to differentiate each function. Simplify the final answer where possible.

- (a) $f(x) = x^2 \sin x$;
- (b) $g(x) = (x^2 + 1) \cos(3x)$;
- (c) $h(x) = \sin x \cos x$.

C8. Use the quotient rule to differentiate the following functions. State where each function is defined.

- (a) $f(x) = \frac{\sin x}{x + 1}$;
- (b) $g(x) = \frac{x}{1 + \cos x}$.

C9. Differentiate each function, using the product, quotient and chain rules as appropriate.

- (a) $y = (x^3 + 1)^2 \sin x$;
- (b) $y = x \cos(x^2)$;
- (c) $y = \frac{\sin x}{\sqrt{x^2 + 1}}$.

C10. A cylindrical can has volume $128\pi \text{ cm}^3$. Let $r > 0$ be its radius and $h > 0$ its height, both in centimetres. The volume and the surface area of a closed cylinder are

$$V = \pi r^2 h, \quad S = 2\pi r^2 + 2\pi r h.$$

- (a) Express h in terms of r , and hence express S as a function of r alone.
- (b) Differentiate $S(r)$ and find its stationary point.
- (c) Show that this stationary point gives a minimum surface area, and find the dimensions of the can.

A1. Let ϕ denote Euler's totient function.

- (a) Calculate $\phi(84)$ and $\phi(100)$ using prime factorisation.
- (b) Use Euler's totient theorem to find the last two digits of 7^{123} . Show how you reduce the exponent.

A2. Consider the affine encryption function

$$f_E(x) \equiv 7x + 3 \pmod{26},$$

with $A = 1$, $B = 2, \dots, Z = 26$, interpreted as elements of $\mathbb{Z}_{26}$.

- (a) Explain why this encryption function has an inverse.
- (b) Use Euclid's algorithm and back-substitution to find $7^{-1} \pmod{26}$, and hence find the decryption function f_D .
- (c) Encrypt the word **CODE**. Decrypt your answer to check your result.

A3. An affine cipher has encryption function

$$f_E(x) \equiv ax + b \pmod{26},$$

where $A = 1, B = 2, \dots, Z = 26$. It is known that the letter A is encrypted as H , and B is encrypted as M .

- (a) Find a and b in $\mathbb{Z}_{26}$.
- (b) Show that the encryption function is invertible, and find the corresponding decryption function f_D .
- (c) Decrypt the ciphertext **QBKKZ**. Check your answer by encrypting the resulting plaintext.