

Sheet 3 – Solutions

MA180 / MA185 / MA190 Mathematics

C1. We have

$$\begin{aligned} f(a+h) &= (a+h)^2 - 3(a+h) + 1 \\ &= a^2 + 2ah + h^2 - 3a - 3h + 1. \end{aligned}$$

Therefore, for $h \neq 0$,

$$\frac{f(a+h) - f(a)}{h} = \frac{2ah + h^2 - 3h}{h} = 2a + h - 3.$$

Taking the limit gives

$$f'(a) = \lim_{h \rightarrow 0} (2a + h - 3) = 2a - 3.$$

At $x = 2$, the point is $(2, -1)$ and the slope is $f'(2) = 1$. Hence the tangent is

$$y = x - 3.$$

C2. For $x \neq 0$ and $x + h \neq 0$,

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h} \left(\frac{4}{x+h} - \frac{4}{x} \right) \\ &= \frac{1}{h} \left(\frac{-4h}{x(x+h)} \right) = -\frac{4}{x(x+h)}. \end{aligned}$$

Thus

$$f'(x) = \lim_{h \rightarrow 0} \left(-\frac{4}{x(x+h)} \right) = -\frac{4}{x^2}, \quad x \neq 0.$$

At $(2, 2)$, the slope is $f'(2) = -1$. The tangent is

$$y = 4 - x.$$

C3. Apply the power rule $\frac{d}{dx} x^n = nx^{n-1}$.

(a)

$$f'(x) = 42x^5 - 12x^2 + 9.$$

(b)

$$g'(x) = 24x^7 + 10x^4 - 2x.$$

(c) Rewrite $h(x) = 5x^{-3} - 2x^{-1} + 4$, so

$$h'(x) = -15x^{-4} + 2x^{-2}.$$

(d) Rewrite $p(x) = 4x^{1/2} - 3x^{2/3}$, so

$$p'(x) = 4 \cdot \frac{1}{2} x^{-1/2} - 3 \cdot \frac{2}{3} x^{-1/3}.$$

C4. (a) Since $f'(x) = 4x^3 - 4x$,

$$f'(-1) = 4(-1)^3 - 4(-1) = 0.$$

(b) Since $g'(x) = -2x^{-3} + 3$,

$$g'(2) = -\frac{2}{2^3} + 3 = \frac{11}{4}.$$

(c) Since $h'(x) = \frac{1}{2\sqrt{x}} + 2x$,

$$h'(4) = \frac{1}{2\sqrt{4}} + 8 = \frac{33}{4}.$$

C5. (a)

$$\begin{aligned} y' &= (2x + 3)(2x^3 - 1) + (x^2 + 3x)(6x^2) \\ &= 10x^4 + 24x^3 - 2x - 3. \end{aligned}$$

(b)

$$\begin{aligned} y' &= 12x^3(x^2 + 5) + (3x^4 - 2)(2x) \\ &= 18x^5 + 60x^3 - 4x. \end{aligned}$$

(c)

$$\begin{aligned} y' &= -x^{-2}(x^3 + 4x) + x^{-1}(3x^2 + 4) \\ &= -x - \frac{4}{x} + 3x + \frac{4}{x} = 2x. \end{aligned}$$

C6. (a)

$$\begin{aligned} f'(x) &= \frac{(x - 2)(2x) - (x^2 + 1)}{(x - 2)^2} \\ &= \frac{x^2 - 4x - 1}{(x - 2)^2}. \end{aligned}$$

(b)

$$\begin{aligned} g'(x) &= \frac{(x^2 + 4)(2) - (2x - 3)(2x)}{(x^2 + 4)^2} \\ &= \frac{-2x^2 + 6x + 8}{(x^2 + 4)^2}. \end{aligned}$$

(c)

$$\begin{aligned} h'(x) &= \frac{(x^2 + 1)(3x^2 - 1) - (x^3 - x)(2x)}{(x^2 + 1)^2} \\ &= \frac{x^4 + 4x^2 - 1}{(x^2 + 1)^2}. \end{aligned}$$

C7. Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2} \\ &= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}. \end{aligned}$$

Substituting $x = 1$ gives

$$f'(1) = \frac{-1 - 2 + 6 + 12 + 6}{(1 + 6)^2} = \frac{21}{49} = \frac{3}{7}.$$

C8. A tangent is horizontal precisely where $y' = 0$. Differentiating,

$$y' = 3x^2 - 6x - 9 = 3(x - 3)(x + 1).$$

Hence $y' = 0$ when $x = -1$ or $x = 3$. Substitution into the original equation gives

$$y(-1) = -1 - 3 + 9 + 5 = 10, \quad y(3) = 27 - 27 - 27 + 5 = -22.$$

Therefore the required points are

$$(-1, 10) \quad \text{and} \quad (3, -22).$$

C9. (a) Since $f(2) = 0$, the difference quotient at $x = 2$ is

$$\frac{f(2+h) - f(2)}{h} = \frac{|h|}{h}, \quad h \neq 0.$$

For $h < 0$, $|h| = -h$, whereas for $h > 0$, $|h| = h$. Consequently,

$$f'_-(2) = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1, \quad f'_+(2) = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1.$$

(b) The one-sided derivatives are different, so the derivative at $x = 2$ does not exist. Thus f is not differentiable at $x = 2$.

(c) Since the absolute-value function is continuous,

$$\lim_{x \rightarrow 2} |x - 2| = 0 = f(2).$$

Hence f is continuous at $x = 2$. There is no contradiction: differentiability implies continuity, but continuity does not imply differentiability.

C10. (a) Take $f(x) = |x|$. Since $\lim_{x \rightarrow 0} |x| = 0 = f(0)$, it is continuous at 0. However, the left-hand derivative is -1 and the right-hand derivative is 1 , so it is not differentiable there.

(b) Take

$$g(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Here $\lim_{x \rightarrow 0} g(x) = 0 \neq g(0) = 1$, so g is not continuous at 0. Since differentiability implies continuity, g cannot be differentiable there.

(c) Take $h(x) = x^2$. The power rule gives $h'(x) = 2x$ for every $x \in \mathbb{R}$, so h is differentiable everywhere.

C11. (a) Differentiate the position function:

$$v(t) = s'(t) = 4t^3 - 12t^2 + 4t + 8.$$

(b) Substituting $t = 1$ and $t = 2$ gives

$$v(1) = 4 - 12 + 4 + 8 = 4 \text{ m/s},$$

$$v(2) = 32 - 48 + 8 + 8 = 0 \text{ m/s}.$$

(c) The particle is at rest when $v(t) = 0$. Factoring,

$$\begin{aligned} 0 &= 4(t^3 - 3t^2 + t + 2) \\ &= 4(t - 2)(t^2 - t - 1). \end{aligned}$$

Thus

$$t = 2 \quad \text{or} \quad t = \frac{1 \pm \sqrt{5}}{2}.$$

Since time after the starting instant must satisfy $t \geq 0$, we discard $(1 - \sqrt{5})/2 < 0$. Therefore the times at rest are

$$t = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad t = 2.$$

- A1. Starting from the right-hand end, double every other digit and subtract 9 whenever a doubled result exceeds 9. The contributions, in their original left-to-right order, are

Digit (positions 1–8)	4	5	3	9	1	4	8	8
Contribution	8	5	6	9	2	4	7	8
Digit (positions 9–16)	0	3	4	3	6	4	6	7
Contribution	0	3	8	3	3	4	3	7

Their sum is

$$S = (8 + 5 + 6 + 9 + 2 + 4 + 7 + 8) + (0 + 3 + 8 + 3 + 3 + 4 + 3 + 7) = 49 + 31 = 80.$$

Since $80 \equiv 0 \pmod{10}$, the number passes Luhn's check.

- A2. The missing digit is in the tenth position from the right, so it is doubled by Luhn's algorithm. The contributions from the known digits are

$$8, 9, 4, 9, 2, 4, \quad ?, 8, 1, 6, 5, 3, 2, 0, 4, 6.$$

They add to 71. Therefore, the missing contribution is 9. The only way a digit can contribute 9 is if it was doubled to get 18. Thus, the original digit is 9.

- A3. For PPS numbers, we have nine (generalized) digits: $d_1, \dots, d_9$. We require that

$$d_8 \equiv 8d_1 + 7d_2 + 6d_3 + 5d_4 + 4d_5 + 3d_6 + 2d_7 + 9d_9 \pmod{23}.$$

(a) For 1234567FA, the sum excluding the check letter F is

$$\begin{aligned} S &= 1(8) + 2(7) + 3(6) + 4(5) + 5(4) + 6(3) + 7(2) + 1(9) \\ &= 8 + 14 + 18 + 20 + 20 + 18 + 14 + 9 = 121. \end{aligned}$$

Now $121 = 23(5) + 6$, and F has value 6. Therefore the check character agrees with the remainder, so 1234567FA passes the check.

(b) The weighted sum is

$$\begin{aligned} S &= 7(8) + 6(7) + 5(6) + 4(5) + 3(4) + 2(3) + 1(2) + 1(9) \\ &= 56 + 42 + 30 + 20 + 12 + 6 + 2 + 9 = 177. \end{aligned}$$

Since $177 = 23(7) + 16$, the remainder is 16. Thus, the missing character with value 16 is P .

A4. An element $a \in \mathbb{Z}_m$ has a multiplicative inverse exactly when $\gcd(a, m) = 1$.

(a) The Euclidean algorithm gives

$$26 = 2(11) + 4, \quad 11 = 2(4) + 3, \quad 4 = 1(3) + 1.$$

Back-substituting,

$$\begin{aligned} 1 &= 4 - 3 = 4 - (11 - 2(4)) \\ &= 3(4) - 11 = 3(26 - 2(11)) - 11 \\ &= 3(26) - 7(11). \end{aligned}$$

Thus $11(-7) \equiv 1 \pmod{26}$, and

$$11^{-1} \equiv 19 \pmod{26}.$$

(b) Since $\gcd(14, 35) = 7$, we know that 14 has no inverse in $\mathbb{Z}_{35}$.

(c) The Euclidean algorithm gives

$$43 = 2(17) + 9, \quad 17 = 1(9) + 8, \quad 9 = 1(8) + 1.$$

Back-substituting,

$$\begin{aligned} 1 &= 9 - 8 = 9 - (17 - 9) \\ &= 2(9) - 17 = 2(43 - 2(17)) - 17 \\ &= 2(43) - 5(17). \end{aligned}$$

Thus $17(-5) \equiv 1 \pmod{43}$, and

$$17^{-1} \equiv 38 \pmod{43}.$$

(d) Since $\gcd(18, 30) \geq 2$, we know that 18 has no inverse in $\mathbb{Z}_{30}$.

A5. A congruence $ax \equiv b \pmod{m}$ has a solution exactly when $\gcd(a, m) \mid b$.

(a) Here $\gcd(15, 21) = 3$, and $3 \mid 9$. Dividing by 3 gives

$$5x \equiv 3 \pmod{7}.$$

Since $5 \cdot 3 \equiv 1 \pmod{7}$, multiplying by 3 yields $x \equiv 2 \pmod{7}$. The solutions in $\mathbb{Z}_{21}$ are

$$x \in \{2, 9, 16\}.$$

(b) Here $\gcd(18, 30) = 6$, and $6 \mid 12$. Dividing by 6 gives

$$3x \equiv 2 \pmod{5}.$$

Since $3 \cdot 2 \equiv 1 \pmod{5}$, we obtain $x \equiv 4 \pmod{5}$. Thus the solutions in $\mathbb{Z}_{30}$ are

$$x \in \{4, 9, 14, 19, 24, 29\}.$$

(c) Since $\gcd(16, 24) = 8$ but $8 \nmid 7$, the congruence has no solutions in $\mathbb{Z}_{24}$.

A6. The moduli 7, 8, 9 are pairwise coprime, so the Chinese Remainder Theorem gives a unique solution modulo

$$M = 7 \cdot 8 \cdot 9 = 504.$$

The partial products are

$$M_1 = \frac{504}{7} = 72, \quad M_2 = \frac{504}{8} = 63, \quad M_3 = \frac{504}{9} = 56.$$

We need inverses of these partial products modulo their corresponding moduli. Since

$$72 \equiv 2 \pmod{7}, \quad 63 \equiv 7 \pmod{8}, \quad 56 \equiv 2 \pmod{9},$$

the inverses are, respectively,

$$2(4) \equiv 1 \pmod{7}, \quad 7(7) \equiv 1 \pmod{8}, \quad 2(5) \equiv 1 \pmod{9}.$$

Thus

$$\begin{aligned} x &\equiv 4M_1(4) + 3M_2(7) + 5M_3(5) \pmod{504} \\ &\equiv 4(72)(4) + 3(63)(7) + 5(56)(5) \\ &\equiv 1152 + 1323 + 1400 = 3875 \\ &\equiv 347 \pmod{504}. \end{aligned}$$

Therefore

$$x = 347 + 504k, \quad k \in \mathbb{Z}.$$

A7. The three lights flash together at intervals of

$$\text{lcm}(6, 7, 11) = 6 \cdot 7 \cdot 11 = 462 \text{ minutes},$$

since 6, 7, 11 are pairwise coprime. Now $462 = 7(60) + 42$, so 462 minutes is 7 hours and 42 minutes.

(a) The next common flash is 7 hours and 42 minutes after noon, namely

$$7:42 \text{ pm}.$$

(b) The interval from noon to 8:00 pm contains 480 minutes. The common-flash times are $462k$ minutes after noon, for non-negative integers k satisfying $462k \leq 480$. Hence $k = 0$ or $k = 1$, giving

$$12:00 \text{ noon and } 7:42 \text{ pm}.$$

A8. (a)

$$756 = 2 \cdot 378 = 2^2 \cdot 189 = 2^2 \cdot 3^3 \cdot 7.$$

(b)

$$1386 = 2 \cdot 693 = 2 \cdot 3^2 \cdot 77 = 2 \cdot 3^2 \cdot 7 \cdot 11.$$

(c)

$$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11.$$

Therefore

$$\gcd(756, 1386) = 2 \cdot 3^2 \cdot 7 = 126,$$

$$\text{lcm}(756, 1386) = 2^2 \cdot 3^3 \cdot 7 \cdot 11 = 8316.$$

A9. (a) **True.** This is Euclid's lemma: a prime that divides a product must divide at least one of its factors.

(b) **False.** Take $n = 6$, $a = 2$ and $b = 3$. Then $6 \mid ab$ because $ab = 6$, but $6 \nmid 2$ and $6 \nmid 3$.

(c) **True.** The only positive divisors of a prime p are 1 and p . If $p \nmid a$, no divisor greater than 1 can be common to a and p . Thus $\gcd(a, p) = 1$.

(d) **True.** By the Fundamental Theorem of Arithmetic, every integer greater than 1 can be written as a product of primes, uniquely up to the order of the factors.

A10. Fermat's little theorem states that, if p is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

(a) Since 13 is prime, $7^{12} \equiv 1 \pmod{13}$. As $100 = 8(12) + 4$,

$$7^{100} \equiv 7^4 \pmod{13}.$$

Now $7^2 = 49 \equiv 10 \pmod{13}$, so

$$7^{100} \equiv 10^2 = 100 \equiv 9 \pmod{13}.$$

(b) Since 17 is prime, $3^{16} \equiv 1 \pmod{17}$. As $222 = 13(16) + 14$,

$$3^{222} \equiv 3^{14} \pmod{17}.$$

But $3^2 = 9$ and $9 \cdot 2 = 18 \equiv 1 \pmod{17}$. Thus multiplying $3^{14} \cdot 3^2 \equiv 3^{16} \equiv 1$ by the inverse of 9 gives

$$3^{222} \equiv 2 \pmod{17}.$$

(c) Since 19 is prime, $11^{18} \equiv 1 \pmod{19}$. As $2026 = 112(18) + 10$,

$$11^{2026} \equiv 11^{10} \pmod{19}.$$

Also $11^2 = 121 \equiv 7 \pmod{19}$ and $11^3 \equiv 7(11) = 77 \equiv 1 \pmod{19}$. Hence $11^{10} = (11^3)^3 \cdot 11 \equiv 11$, giving

$$11^{2026} \equiv 11 \pmod{19}.$$

A11. (a) The congruence can be written as

$$(x - 2)(x + 2) \equiv 0 \pmod{11}.$$

Since 11 is prime, it divides $x - 2$ or $x + 2$ by Euclid's Lemma. Therefore the solutions in $\mathbb{Z}_{11}$ are

$$x \in \{2, 9\}.$$

(b) We have $6^2 = 36 \equiv 10 \pmod{13}$, so $x = 6$ is a solution. Since

$$x^2 - 36 = (x - 6)(x + 6),$$

and 13 is prime, every solution satisfies $x \equiv 6$ or $x \equiv -6 \pmod{13}$. Thus

$$x \in \{6, 7\}.$$

(c) Note that $x = 2$ is a solution, since $2^3 = 8$. To see that it is the only one, note that $x^3 \equiv 8 \pmod{17}$ implies $x \not\equiv 0 \pmod{17}$. By Fermat's little theorem, $x^{16} \equiv 1 \pmod{17}$. Since $33 = 2(16) + 1$, raising the original congruence to the eleventh power gives

$$x^{33} \equiv 8^{11} = 2^{33} \pmod{17}.$$

But $x^{33} = x(x^{16})^2 \equiv x$, and similarly $2^{33} \equiv 2 \pmod{17}$. Hence

$$x = 2 \quad \text{in } \mathbb{Z}_{17}$$

is the unique solution.