

Sheet 3

MA180 / MA185 / MA190 Mathematics

C1. Let $f(x) = x^2 - 3x + 1$. Use the definition

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

to find $f'(a)$. Hence find the equation of the tangent to the graph of $y = f(x)$ at the point where $x = 2$.

C2. Let $f(x) = \frac{4}{x}$. Using the limit definition of the derivative, find $f'(x)$ for $x \neq 0$. Hence find the equation of the tangent to $y = 4/x$ at $(2, 2)$.

C3. Differentiate each function.

(a) $f(x) = 7x^6 - 4x^3 + 9x - 5$;

(b) $g(x) = 3x^8 + 2x^5 - x^2 + 6$;

(c) $h(x) = \frac{5}{x^3} - \frac{2}{x} + 4$;

(d) $p(x) = 4\sqrt{x} - 3\sqrt[3]{x^2}$.

C4. Find the value of each derivative at the stated point.

(a) $f(x) = x^4 - 2x^2 + 3$, at $x = -1$;

(b) $g(x) = x^{-2} + 3x$, at $x = 2$;

(c) $h(x) = \sqrt{x} + x^2$, at $x = 4$.

C5. Use the product rule to differentiate each function. Simplify the answer.

(a) $y = (x^2 + 3x)(2x^3 - 1)$;

(b) $y = (3x^4 - 2)(x^2 + 5)$;

(c) $y = x^{-1}(x^3 + 4x)$.

C6. Use the quotient rule to differentiate each function. State any restrictions on x .

(a) $f(x) = \frac{x^2 + 1}{x - 2}$;

(b) $g(x) = \frac{2x - 3}{x^2 + 4}$;

(c) $h(x) = \frac{x^3 - x}{x^2 + 1}$.

C7. Let

$$f(x) = \frac{x^2 + x - 2}{x^3 + 6}.$$

Differentiate f using the quotient rule. Then calculate $f'(1)$.

C8. Find all points on the graph

$$y = x^3 - 3x^2 - 9x + 5$$

at which the tangent is horizontal.

C9. Consider $f(x) = |x - 2|$.

- (a) Use one-sided difference quotients to find the left-hand and right-hand derivatives at $x = 2$.
- (b) Is f differentiable at $x = 2$?
- (c) Is f continuous at $x = 2$? Explain why this does not contradict the theorem that differentiability implies continuity.

C10. Give an example of each of the following, with a brief explanation.

- (a) A function continuous at $x = 0$ but not differentiable there.
- (b) A function that is not continuous at $x = 0$ and therefore cannot be differentiable there.
- (c) A function differentiable for every real x .

C11. A particle has position

$$s(t) = t^4 - 4t^3 + 2t^2 + 8t$$

metres after t seconds.

- (a) Find the velocity function $v(t) = s'(t)$.
- (b) Find the velocity at $t = 1$ and $t = 2$.
- (c) Find all times at which the particle is instantaneously at rest.

A1. Apply Luhn's algorithm to the number

4539 1488 0343 6467.

Show the contribution from each digit and determine whether the number passes the check.

A2. One digit is missing from

4929 14?8 5673 1026.

Use Luhn's algorithm to determine the missing digit.

A3. (a) Verify that 1234567FA is a valid PPS number (i.e., it passes the check).

(b) Find the check character in the PPS number 7654321?A.

A4. Determine whether each element has a multiplicative inverse in the stated residue system. If it exists, compute it using Euclid's algorithm and back-substitution.

- (a) 11 in $\mathbb{Z}_{26}$;
- (b) 14 in $\mathbb{Z}_{35}$;

- (c) 17 in $\mathbb{Z}_{43}$;
- (d) 18 in $\mathbb{Z}_{30}$.

A5. Solve each congruence in the stated residue system, listing all solutions.

- (a) $15x \equiv 9 \pmod{21}$, $x \in \mathbb{Z}_{21}$;
- (b) $18x \equiv 12 \pmod{30}$, $x \in \mathbb{Z}_{30}$;
- (c) $16x \equiv 7 \pmod{24}$, $x \in \mathbb{Z}_{24}$.

A6. Use the Chinese Remainder Theorem to solve

$$\begin{cases} x \equiv 4 \pmod{7}, \\ x \equiv 3 \pmod{8}, \\ x \equiv 5 \pmod{9}. \end{cases}$$

Show the partial products and the modular inverses used.

A7. Three warning lights flash every 6, 7, and 11 minutes, respectively. They flash together at noon.

- (a) When will they next flash together?
- (b) At what times between noon and 8:00 p.m., inclusive, will all three flash together?

A8. Find the prime-power factorisation of each integer.

- (a) 756;
- (b) 1386;
- (c) 2310.

Hence compute $\gcd(756, 1386)$ and $\text{lcm}(756, 1386)$.

A9. State whether each assertion is true or false, and justify the answer.

- (a) If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.
- (b) If $n \mid ab$, then $n \mid a$ or $n \mid b$ for every positive integer n .
- (c) If p is prime and $p \nmid a$, then $\gcd(a, p) = 1$.
- (d) Every integer greater than 1 has a unique prime factorisation, apart from the order of the factors.

A10. Use Fermat's little theorem to compute the following standard representatives.

- (a) $7^{100} \pmod{13}$;
- (b) $3^{222} \pmod{17}$;
- (c) $11^{2026} \pmod{19}$.

A11. Find all solutions in the stated residue system.

- (a) $x^2 \equiv 4 \pmod{11}$, $x \in \mathbb{Z}_{11}$;
- (b) $x^2 \equiv 10 \pmod{13}$, $x \in \mathbb{Z}_{13}$;
- (c) $x^3 \equiv 8 \pmod{17}$, $x \in \mathbb{Z}_{17}$.