

## Sheet 2 – Solutions

MA180 / MA185 / MA190 Mathematics

C1. (a) Since polynomials are continuous everywhere,

$$\lim_{x \rightarrow 2} (3x^3 - 5x + 4) = 3(2)^3 - 5(2) + 4 = 24 - 10 + 4 = 18.$$

(b) The numerator and denominator are polynomials, and the denominator is non-zero at  $x = -1$  since  $(-1)^2 + 4 = 5$ . Hence the quotient law applies:

$$\lim_{x \rightarrow -1} \frac{x^2 + 2x + 5}{x^2 + 4} = \frac{1 - 2 + 5}{1 + 4} = \frac{4}{5}.$$

(c) Since  $\sin x$  and  $\cos x$  are continuous,

$$\lim_{x \rightarrow \pi/3} (2 \sin x + \cos x) = 2 \sin \frac{\pi}{3} + \cos \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} = \sqrt{3} + \frac{1}{2}.$$

C2. (a) Factor the numerator:

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 \quad (x \neq 3).$$

Therefore

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6.$$

(b) Since  $x^2 + 5x + 6 = (x + 2)(x + 3)$ ,

$$\frac{x^2 + 5x + 6}{x + 2} = x + 3 \quad (x \neq -2),$$

so

$$\lim_{x \rightarrow -2} \frac{x^2 + 5x + 6}{x + 2} = 1.$$

(c) Rationalising the numerator gives

$$\frac{\sqrt{x+4}-3}{x-5} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \frac{x+4-9}{(x-5)(\sqrt{x+4}+3)} = \frac{1}{\sqrt{x+4}+3}$$

for  $x \neq 5$ . Hence

$$\lim_{x \rightarrow 5} \frac{\sqrt{x+4}-3}{x-5} = \frac{1}{3+3} = \frac{1}{6}.$$

(d) Rationalising gives

$$\frac{\sqrt{1+3x}-1}{x} \cdot \frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} = \frac{3x}{x(\sqrt{1+3x}+1)} = \frac{3}{\sqrt{1+3x}+1}$$

for  $x \neq 0$ . Therefore

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x}-1}{x} = \frac{3}{2}.$$

C3. If  $x < 2$  and  $x$  is sufficiently close to 2, then  $1 \leq x < 2$ , so  $\lfloor x \rfloor = 1$ . Thus

$$\lim_{x \rightarrow 2^-} \lfloor x \rfloor = 1.$$

If  $x > 2$  and  $x$  is sufficiently close to 2, then  $2 \leq x < 3$ , so  $\lfloor x \rfloor = 2$ . Thus

$$\lim_{x \rightarrow 2^+} \lfloor x \rfloor = 2.$$

Since the one-sided limits are different,  $\lim_{x \rightarrow 2} \lfloor x \rfloor$  does not exist.

More generally, for any integer  $n$ ,

$$\lim_{x \rightarrow n^-} \lfloor x \rfloor = n - 1, \quad \lim_{x \rightarrow n^+} \lfloor x \rfloor = n.$$

Hence the two-sided limit does not exist at any integer  $n$ .

C4. For  $x < -1$ , we have  $x + 1 < 0$ , so  $|x + 1| = -(x + 1)$  and therefore

$$g(x) = \frac{x + 1}{-(x + 1)} = -1.$$

For  $x > -1$ , we have  $x + 1 > 0$ , so

$$g(x) = \frac{x + 1}{x + 1} = 1.$$

Hence

$$\lim_{x \rightarrow -1^-} g(x) = -1, \quad \lim_{x \rightarrow -1^+} g(x) = 1.$$

The graph consists of the horizontal line  $y = -1$  for  $x < -1$  and the horizontal line  $y = 1$  for  $x > -1$ , with open circles at  $(-1, -1)$  and  $(-1, 1)$ . Since the one-sided limits differ, the two-sided limit at  $-1$  does not exist.

C5. As  $x \rightarrow 0^+$ , the number  $x$  is positive and can be made arbitrarily small, so  $1/x$  becomes arbitrarily large and positive:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty.$$

As  $x \rightarrow 0^-$ , the number  $x$  is negative with arbitrarily small magnitude, so  $1/x$  becomes arbitrarily large in magnitude and negative:

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

The one-sided behaviours are different, so there is no finite real number  $L$  for which  $\lim_{x \rightarrow 0} 1/x = L$ . Thus the two-sided limit does not exist as a finite real number.

C6. A function  $f$  is continuous at  $x = a$  if all three of the following hold:

- (i)  $f(a)$  is defined;
- (ii)  $\lim_{x \rightarrow a} f(x)$  exists;
- (iii)  $\lim_{x \rightarrow a} f(x) = f(a)$ .

Here, for  $x \neq 1$ ,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1.$$

Hence

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x + 1) = 2.$$

But  $f(1) = 3$ . Since  $\lim_{x \rightarrow 1} f(x) \neq f(1)$ , the function is not continuous at  $x = 1$ .

C7. For  $x \neq 2$ ,

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2.$$

Therefore

$$\lim_{x \rightarrow 2} f(x) = 4.$$

Continuity at  $x = 2$  requires  $f(2) = \lim_{x \rightarrow 2} f(x)$ , so

$$\boxed{k = 4}.$$

C8. Continuity at  $x = 1$  requires the left-hand value to match the middle piece:

$$a + b = 1^2 + 1 = 2.$$

Continuity at  $x = 3$  requires

$$\lim_{x \rightarrow 3^-} (x^2 + 1) = f(3).$$

Thus

$$10 = 4(3) - a = 12 - a,$$

so  $a = 2$ . Substituting into  $a + b = 2$  gives  $b = 0$ . Therefore

$$\boxed{a = 2, \quad b = 0}.$$

C9. Factor numerator and denominator:

$$f(x) = \frac{(x - 1)(x + 1)}{(x - 2)(x + 1)}.$$

The original denominator vanishes at  $x = -1$  and  $x = 2$ , so these are the only possible discontinuities. For  $x \neq -1, 2$ ,

$$f(x) = \frac{x - 1}{x - 2}.$$

At  $x = -1$ ,

$$\lim_{x \rightarrow -1} f(x) = \frac{-2}{-3} = \frac{2}{3}.$$

The limit exists and is finite, but the original function is undefined at  $x = -1$ . Hence  $x = -1$  is a removable discontinuity.

At  $x = 2$ , the simplified numerator tends to 1, while the denominator tends to 0. Moreover,

$$\lim_{x \rightarrow 2^-} \frac{x - 1}{x - 2} = -\infty, \quad \lim_{x \rightarrow 2^+} \frac{x - 1}{x - 2} = +\infty.$$

Thus  $x = 2$  is a non-removable discontinuity (a vertical asymptote).

C10. Let  $\varepsilon > 0$ . We want

$$|(5x - 1) - 9| < \varepsilon.$$

But

$$|(5x - 1) - 9| = |5x - 10| = 5|x - 2|.$$

Choose

$$\boxed{\delta = \frac{\varepsilon}{5}}.$$

Then whenever  $0 < |x - 2| < \delta$ ,

$$|(5x - 1) - 9| = 5|x - 2| < 5\delta = \varepsilon.$$

Therefore, by the  $\varepsilon$ - $\delta$  definition,

$$\lim_{x \rightarrow 2} (5x - 1) = 9.$$

C11. Let  $\varepsilon > 0$ . We have

$$|(3 - 2x) - 5| = |-2x - 2| = 2|x + 1|.$$

Choose

$$\boxed{\delta = \frac{\varepsilon}{2}}.$$

If  $0 < |x + 1| < \delta$ , then

$$|(3 - 2x) - 5| = 2|x + 1| < 2\delta = \varepsilon.$$

Hence

$$\lim_{x \rightarrow -1} (3 - 2x) = 5.$$

C12. Let  $\varepsilon > 0$ . We need to make

$$|x^2 - 4| = |x - 2||x + 2|$$

small. First require  $|x - 2| < 1$ . Then

$$1 < x < 3,$$

and hence

$$3 < x + 2 < 5,$$

so  $|x + 2| < 5$ .

Choose

$$\boxed{\delta = \min \left\{ 1, \frac{\varepsilon}{5} \right\}}.$$

If  $0 < |x - 2| < \delta$ , then  $|x - 2| < 1$  and therefore  $|x + 2| < 5$ . Thus

$$|x^2 - 4| = |x - 2||x + 2| < 5|x - 2| < 5\delta \leq \varepsilon.$$

Therefore

$$\lim_{x \rightarrow 2} x^2 = 4.$$

---

A1. (a) By the Euclidean algorithm,

$$35 = 1 \cdot 24 + 11, \quad 24 = 2 \cdot 11 + 2, \quad 11 = 5 \cdot 2 + 1.$$

Hence  $\gcd(24, 35) = 1$ , so 24 and 35 are coprime.

(b)

$$70 = 1 \cdot 42 + 28, \quad 42 = 1 \cdot 28 + 14, \quad 28 = 2 \cdot 14.$$

Hence  $\gcd(42, 70) = 14$ , so they are not coprime.

(c)

$$105 = 1 \cdot 64 + 41, \quad 64 = 1 \cdot 41 + 23, \quad 41 = 1 \cdot 23 + 18,$$

$$23 = 1 \cdot 18 + 5, \quad 18 = 3 \cdot 5 + 3, \quad 5 = 1 \cdot 3 + 2, \quad 3 = 1 \cdot 2 + 1.$$

Hence  $\gcd(105, 64) = 1$ , so 105 and 64 are coprime.

A2. (a)

$$\boxed{137 = 12 \cdot 11 + 5}.$$

(b) Since the remainder must satisfy  $0 \leq r < 7$ ,

$$\boxed{-53 = 7(-8) + 3}.$$

(c)

$$\boxed{2026 = 23 \cdot 88 + 2}.$$

A3. (a)  $83 = 7 \cdot 11 + 6$ , so

$$\boxed{83 \equiv 6 \pmod{7}}.$$

(b)  $-41 = 9(-5) + 4$ , so

$$\boxed{-41 \equiv 4 \pmod{9}}.$$

(c)  $5 + 17 = 22$  and  $22 = 8 \cdot 2 + 6$ , so

$$\boxed{5 + 17 \equiv 6 \pmod{8}}.$$

(d)  $7 \cdot 13 = 91$  and  $91 = 10 \cdot 9 + 1$ , so

$$\boxed{7 \cdot 13 \equiv 1 \pmod{10}}.$$

A4. Recall that  $a \equiv b \pmod{m}$  exactly when  $m \mid (a - b)$ .

(a)  $47 - 5 = 42 = 7 \cdot 6$ , so

$$\boxed{47 \equiv 5 \pmod{7}}.$$

(b)  $-8 - 4 = -12 = 6(-2)$ , so

$$\boxed{-8 \equiv 4 \pmod{6}}.$$

(c)  $31 - 2 = 29$ , and  $10 \nmid 29$ , so

$$\boxed{31 \not\equiv 2 \pmod{10}}.$$

(d)  $1001 = 7 \cdot 143$ , so  $7 \mid 1001$  and hence

$$\boxed{1001 \equiv 0 \pmod{7}}.$$

A5. An equation  $ax + by = c$  has an integer solution if and only if  $\gcd(a, b) \mid c$ .

(a) Since  $\gcd(18, 30) = 6$  and  $6 \nmid 7$ , there is no integer solution.

(b) Since  $6 \mid 6$ , integer solutions exist. For example,

$$18(2) + 30(-1) = 36 - 30 = 6,$$

so one solution is

$$\boxed{(x, y) = (2, -1)}.$$

(c) Since  $\gcd(35, 64) = 1$ , a solution exists. Indeed,

$$35(11) + 64(-6) = 385 - 384 = 1,$$

so one solution is

$$\boxed{(x, y) = (11, -6)}.$$

A6. (a) Since  $\gcd(414, 662) = 2$ , divide by 2:

$$207x + 331y = 1.$$

One solution of the original equation is  $(x_0, y_0) = (8, -5)$ , since

$$414(8) + 662(-5) = 3312 - 3310 = 2.$$

Therefore all integer solutions are

$$\boxed{x = 8 + 331t, \quad y = -5 - 207t, \quad t \in \mathbb{Z}.}$$

(b) Divide by  $\gcd(21, 15) = 3$ :

$$7x + 5y = 4.$$

A particular solution is  $(2, -2)$ . Hence all integer solutions are

$$\boxed{x = 2 + 5t, \quad y = -2 - 7t, \quad t \in \mathbb{Z}.}$$

(c) Since  $\gcd(105, 64) = 1$ , solutions exist. A particular solution is

$$105(25) + 64(-41) = 2625 - 2624 = 1.$$

Hence all integer solutions are

$$\boxed{x = 25 + 64t, \quad y = -41 - 105t, \quad t \in \mathbb{Z}.}$$

A7. We solve

$$7s + 11t = 173, \quad s, t \geq 0.$$

Reducing modulo 7 gives

$$11t \equiv 173 \pmod{7},$$

so

$$4t \equiv 5 \pmod{7}.$$

Since  $2 \cdot 4 \equiv 1 \pmod{7}$ , multiplying by 2 gives

$$t \equiv 3 \pmod{7}.$$

Thus  $t = 3 + 7k$  for some  $k \in \mathbb{Z}$ . Since  $t \geq 0$  and  $11t \leq 173$ , we have  $0 \leq t \leq 15$ , so the only possibilities are  $t = 3$  and  $t = 10$ .

If  $t = 3$ , then

$$s = \frac{173 - 33}{7} = 20.$$

If  $t = 10$ , then

$$s = \frac{173 - 110}{7} = 9.$$

Therefore

(a) the non-negative integer solutions are

$$\boxed{(s, t) = (20, 3) \quad \text{or} \quad (9, 10)};$$

(b) either the cinema sold 20 student tickets and 3 standard tickets, or it sold 9 student tickets and 10 standard tickets.

A8. Suppose  $d \mid a$  and  $d \mid b$ . Then there exist integers  $k$  and  $\ell$  such that

$$a = dk, \quad b = d\ell.$$

For arbitrary  $r, s \in \mathbb{Z}$ ,

$$ra + sb = r(dk) + s(d\ell) = d(rk + s\ell).$$

Since  $rk + s\ell \in \mathbb{Z}$ , it follows from the definition of divisibility that

$$\boxed{d \mid (ra + sb)}.$$

A9. Assume

$$a \equiv b \pmod{m} \quad \text{and} \quad c \equiv d \pmod{m}.$$

Then  $m \mid (a - b)$  and  $m \mid (c - d)$ . Hence there exist  $r, s \in \mathbb{Z}$  such that

$$a - b = mr, \quad c - d = ms.$$

Now

$$ac - bd = ac - bc + bc - bd = c(a - b) + b(c - d).$$

Substituting the expressions above,

$$ac - bd = c(mr) + b(ms) = m(cr + bs).$$

Since  $cr + bs \in \mathbb{Z}$ , we have  $m \mid (ac - bd)$ . Therefore

$$\boxed{ac \equiv bd \pmod{m}}.$$