Sheet 2

MA180 / MA185 / MA190 Mathematics

C1. Evaluate the following limits by direct substitution, explaining why the relevant limit rules apply:

(a) $\lim_{x\to 2}(3x^3 - 5x + 4)$;

(b) $\lim_{x\to -1} \dfrac{x^2 + 2x + 5}{x^2 + 4}$;

(c) $\lim_{x\to \pi/3}(2\sin x + \cos x)$.

C2. Evaluate each limit:

(a) $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3}$;

(b) $\lim_{x\to -2} \dfrac{x^2 + 5x + 6}{x + 2}$;

(c) $\lim_{x\to 5} \dfrac{\sqrt{x+4} - 3}{x - 5}$;

(d) $\lim_{x\to 0} \dfrac{\sqrt{1+3x} - 1}{x}$.

C3. Let $\lfloor x \rfloor$ denote the floor function. Find

$$\lim_{x\to 2^-} \lfloor x \rfloor \qquad \text{and} \qquad \lim_{x\to 2^+} \lfloor x \rfloor.$$

Does $\lim_{x\to 2} \lfloor x \rfloor$ exist? Generalise the result to an arbitrary integer n.

C4. Consider

$$g(x) = \frac{x+1}{|x+1|}, \qquad x \neq -1.$$

Find the left-hand and right-hand limits at $x = -1$, and sketch enough of the graph to explain the answer.

C5. Explain carefully why $\lim_{x\to 0} 1/x$ does not exist as a finite real number. State the behaviour as $x \to 0^+$ and as $x \to 0^-$.

C6. State the three conditions required for a function f to be continuous at $x = a$. Then decide whether

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1, \\ 3, & x = 1 \end{cases}$$

is continuous at $x = 1$.

C7. Find the value of k that makes the function continuous at $x = 2$:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2, \\ k, & x = 2. \end{cases}$$

C8. Find constants a and b so that

$$f(x) = \begin{cases} ax + b, & x < 1, \\ x^2 + 1, & 1 \leq x < 3, \\ 4x - a, & x \geq 3 \end{cases}$$

is continuous on $\mathbb{R}$.

C9. Determine all points of discontinuity of

$$f(x) = \frac{x^2 - 1}{x^2 - x - 2}.$$

Classify each discontinuity as removable or non-removable, and justify the classification using limits.

C10. Using the ε–δ definition, prove that

$$\lim_{x \to 2}(5x - 1) = 9.$$

Give an explicit choice of δ in terms of ε.

C11. Using the ε–δ definition, prove that

$$\lim_{x \to -1}(3 - 2x) = 5.$$

C12. Prove directly from the ε–δ definition that

$$\lim_{x \to 2} x^2 = 4.$$

Hint: first require $|x - 2| < 1$ and use this to bound $|x + 2|$.

A1. State whether each pair is coprime. Justify each answer by computing the gcd.

(a) 24 and 35;

(b) 42 and 70;

(c) 105 and 64.

A2. Use the remainder theorem to write $a = bq + r$, with $0 \leq r < b$, in each case.

(a) $a = 137$, $b = 12$;

(b) $a = -53$, $b = 7$;

(c) $a = 2026$, $b = 23$.

A3. Compute the following standard representatives.

(a) $83 \pmod 7$;

(b) $-41 \pmod 9$;

(c) $5 + 17 \pmod 8$;

(d) $7 \cdot 13 \pmod{10}$.

A4. For each congruence, either verify it directly from the definition or show that it is false.

(a) $47 \equiv 5 \pmod{7}$;

(b) $-8 \equiv 4 \pmod{6}$;

(c) $31 \equiv 2 \pmod{10}$;

(d) $1001 \equiv 0 \pmod{7}$.

A5. Determine whether each equation has an integer solution. Give a reason; if a solution exists, find one.

(a) $18x + 30y = 7$;

(b) $18x + 30y = 6$;

(c) $35x + 64y = 1$.

A6. Find all integer solutions (x, y) for each equation.

(a) $414x + 662y = 2$;

(b) $21x + 15y = 12$;

(c) $105x + 64y = 1$.

A7. A cinema sells student tickets for 7 euro and standard tickets for 11 euro. At one screening, ticket revenue was 173 euro.

(a) Find all non-negative integer solutions of $7s + 11t = 173$.

(b) Interpret the solutions in the context of the problem.

A8. Prove that if $d \mid a$ and $d \mid b$, then $d \mid (ra + sb)$ for every $r, s \in \mathbb{Z}$.

A9. Prove the following compatibility property of congruences: if

$$a \equiv b \pmod{m} \quad \text{and} \quad c \equiv d \pmod{m},$$

then

$$ac \equiv bd \pmod{m}.$$