Additional Practice: Linear Algebra and Matrix Algebra
Solutions

MA180/MA185/MA190

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1 Linear systems and modelling

1. (a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & 1 & -3 & -4 \\ -1 & 4 & 2 & 9 \end{array}\right).$$

(b) In $Av = b$,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 4 & 2 \end{pmatrix}, \qquad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix}.$$

(c) A is 3×3, and v and b are both 3×1.

2. (a) Substituting the three points gives

$$c = 2, \qquad a + b + c = 1, \qquad a - b + c = 5.$$

(b) With $c = 2$, we have $a + b = -1$ and $a - b = 3$. Adding gives $2a = 2$, so $a = 1$ and $b = -2$. Thus $y = x^2 - 2x + 2$.

(c) For $f(x) = x^2 - 2x + 2$, we have $f(0) = 2$, $f(1) = 1 - 2 + 2 = 1$, and $f(-1) = 1 + 2 + 2 = 5$.

3. (a) The fat, protein, and carbohydrate requirements give

$$2x + y + 3z = 14, \qquad x + 2y + z = 7, \qquad 3x + 2y + 4z = 20.$$

(b) In $Av = b$,

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 2 & 4 \end{pmatrix}, \qquad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad b = \begin{pmatrix} 14 \\ 7 \\ 20 \end{pmatrix}.$$

Swap R_1 and R_2, then use $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 3R_1$. Finally use $R_3 \leftarrow R_3 - \frac{4}{3}R_2$:

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 14 \\ 1 & 2 & 1 & 7 \\ 3 & 2 & 4 & 20 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -3 & 1 & 0 \\ 0 & -4 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right).$$

Back substitution gives $z = 3$, $y = 1$, and $x = 2$.

(c) Use two portions of X, one of Y, and three of Z. These are 200, 100, and 300 grams, respectively.

2 Row operations and Gaussian elimination

4. The three steps are

$$\left(\begin{array}{cc|c} 0 & 2 & 4 \\ 1 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 4 \end{array}\right) \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right).$$

The operations are a row interchange, multiplication of a row by a non-zero scalar, and addition of a multiple of one row to another. The solution is $(x, y) = (1, 2)$.

5. Use $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 + R_1$, then $R_3 \leftarrow R_3 - 3R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 5 & 1 & 10 \\ -1 & 1 & 2 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 10 \\ 0 & 3 & 1 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & -8 & -24 \end{array}\right).$$

The final matrix is in REF. Back substitution gives $z = 3$, $y = 10 - 3z = 1$, and $x = z - 2y = 1$. Substitution in the original equations gives

$$1 + 2 - 3 = 0, \qquad 2 + 5 + 3 = 10, \qquad -1 + 1 + 6 = 6.$$

6. Swap R_1 and R_2 to put a non-zero entry in the first pivot position. Then use $R_3 \leftarrow R_3 - 2R_1$, followed by $R_3 \leftarrow R_3 - \frac{3}{2}R_2$:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 1 & -1 & 2 & 6 \\ 2 & 1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & -3 & -9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -\frac{9}{2} & -9 \end{array}\right).$$

From the REF, $z = 2$, then $2y = -2$ gives $y = -1$, and $x = 6 + y - 2z = 1$. Thus $(x, y, z) = (1, -1, 2)$.

7. Apply $R_1 \leftarrow R_1 - 2R_2$ to obtain the RREF

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   (a) The equations are $x - 5z = -5$ and $y + 2z = 4$, with the last row giving $0 = 0$.

   (b) Taking $z = u$ gives all solutions $(x, y, z) = (5u - 5, 4 - 2u, u)$ for $u \in \mathbb{R}$.

   (c) The choices $u = 0$ and $u = 1$ give $(-5, 4, 0)$ and $(0, 2, 1)$. Each real u gives a distinct solution because its third coordinate is u.

8.
- A is in REF only. Its pivots are $1, 1, 3$ at $(1, 1), (2, 2), (3, 3)$. They move strictly to the right down the rows, with zeros below each pivot. The third pivot is 3, and the second and third pivot columns have non-zero entries above their pivots.
- B is in RREF and hence in REF. Its pivots are the ones at $(1, 1)$ and $(2, 2)$. Each pivot is the only non-zero entry in its column, and the zero row is at the bottom.
- C is in neither form. Its first two rows both lead in column 1. Applying $R_2 \leftarrow R_2 - R_1$ and then $R_3 \leftarrow R_3 + \frac{1}{2}R_2$ gives an echelon matrix with pivots $1, -2, \frac{3}{2}$ at $(1, 1), (2, 2), (3, 3)$.

3 Number of solutions of linear systems

9. (a) Unique solution. The second row gives $2y = 4$, and the first then gives $x + 3(2) = 7$. Thus $(x, y) = (1, 2)$.

   (b) No solution. The second row gives $0 = 3$.

   (c) Infinitely many solutions. The last row gives $0 = 0$, and z is free. The other rows give $y = 2 - 4z$ and $x = -1 + 9z$.

10. Apply $R_2 \leftarrow R_2 - 2R_1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & t & b \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & t-2 & b-6 \end{array}\right).$$

   (a) If $t \neq 2$, there is a unique solution:

$$y = \frac{b-6}{t-2}, \qquad x = 3 - y = \frac{3t-b}{t-2}.$$

   (b) If $t = 2$ and $b = 6$, the last row gives $0 = 0$. Any real y with $x = 3 - y$ is a solution, so there are infinitely many solutions.

   (c) If $t = 2$ and $b \neq 6$, the last row gives $0 = b - 6$, a contradiction. There is no solution.

11. Subtract the original first and second rows from the third, then apply $R_2 \leftarrow R_2 - R_1$. The resulting augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & t & s \end{array}\right).$$

(a) If $t \neq 0$, for any s there is a unique solution. The third row gives $z = s/t$, the second gives $y = 4 - 2z$, and the first gives $x = z - 1$.

(b) If $t = 0$ and $s \neq 0$, the last row gives $0 = s$, so there is no solution.

(c) If $t = s = 0$, the last row gives $0 = 0$. Every real z gives a solution $(z - 1, 4 - 2z, z)$, so there are infinitely many solutions.

4 Matrix dimensions, entries, and arithmetic

12. (a) M is 2×4.

(b) $M_{14} = 4$, $M_{21} = 3$, $M_{23} = -1$, and $M_{24} = 5$.

(c) $-3M$ is 2×4.

(d) Only $M + P$ is defined, since matrix addition requires matching dimensions and P is also 2×4.

13. (a)
$$P + Q = \begin{pmatrix} 1+0 & -2+5 & 0+2 \\ 3-1 & 1+3 & 4+1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \end{pmatrix}.$$

(b) $P - Q = \begin{pmatrix} 1 & -7 & -2 \\ 4 & -2 & 3 \end{pmatrix}$.

(c) $-2P + 3Q = \begin{pmatrix} -2 & 19 & 6 \\ -9 & 7 & -5 \end{pmatrix}$.

14. Rearranging gives
$$X = \tfrac{1}{2}(3B - A) = \tfrac{1}{2}\begin{pmatrix} -2 & 8 \\ 14 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 7 & -\frac{3}{2} \end{pmatrix}.$$

Substitution gives
$$2X + A = \begin{pmatrix} -2 & 8 \\ 14 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 12 & -3 \end{pmatrix} = 3B.$$

For $2X + A$ to be defined, $2X$ must have the dimensions of A. Scalar multiplication preserves dimensions, and A and B are both 2×2.

5 Matrix multiplication and column vectors

15. The product is defined when the inner dimensions agree. The outer dimensions then give the size of the result.

Product	Factor dimensions	Result
AB	$(2 \times 3)(3 \times 2)$	2×2
BA	$(3 \times 2)(2 \times 3)$	3×3
AC	$(2 \times 3)(2 \times 2)$	undefined
CA	$(2 \times 2)(2 \times 3)$	2×3
BC	$(3 \times 2)(2 \times 2)$	3×2
CD	$(2 \times 2)(3 \times 1)$	undefined
DA	$(3 \times 1)(2 \times 3)$	undefined

16. (a) Row-column multiplication gives

$$AB = \begin{pmatrix} 1\cdot 2 + 2\cdot 3 & 1\cdot 0 + 2\cdot 1 \\ 0\cdot 2 + 1\cdot 3 & 0\cdot 0 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 3 & 1 \end{pmatrix},$$
$$BA = \begin{pmatrix} 2\cdot 1 + 0\cdot 0 & 2\cdot 2 + 0\cdot 1 \\ 3\cdot 1 + 1\cdot 0 & 3\cdot 2 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}.$$

(b) The products differ, for example in their $(1,1)$ entries.

(c) Matrix multiplication is not commutative in general, even for two square matrices of the same size.

17. (a) $2x + y = 8$ and $x - y = 1$.

(b) Swap the rows, then eliminate the entry below the first pivot:

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & -1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 6 \end{array}\right).$$

Thus $y = 2$ and $x = 1 + y = 3$.

(c)

$$Av = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + 2 \\ 3 - 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} = b.$$

6 Matrix identities and special matrices

18. (a) True. Each entry of $A + B$ is $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.

(b) False. The matrices in Question 16 are a counterexample, with $(AB)_{11} = 8$ and $(BA)_{11} = 2$.

(c) True. For each entry,

$$[A(B + C)]_{ij} = \sum_k a_{ik}(b_{kj} + c_{kj}) = \sum_k a_{ik}b_{kj} + \sum_k a_{ik}c_{kj} = [AB + AC]_{ij}.$$

(d) False. Take $A = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then AB is 1×3, but the inner dimensions of BA are 3 and 1.

(e) True. Every entry of Z is zero, so $(A + Z)_{ij} = a_{ij}$. The zero matrix is the additive identity for matrices of its size.

(f) False. The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is diagonal because its off-diagonal entries are zero. The identity matrix has diagonal entries all equal to 1 and acts as the multiplicative identity.

7 Determinants and inverses of 2×2 matrices

19. (a) $\det(A) = 3 \cdot 1 - 2 \cdot 1 = 1$ and $\det(B) = 4 \cdot 3 - 6 \cdot 2 = 0$.

(b) A is invertible because its determinant is non-zero. B is singular because its determinant is zero.

(c) $\det(C_t) = t(t-1) - 6 = (t-3)(t+2)$, so C_t is singular exactly when $t = 3$ or $t = -2$.

(d) C_t is invertible for every $t \in \mathbb{R} \setminus \{-2, 3\}$.

20. (a)
$$AB = \begin{pmatrix} 2p+r & 2q+s \\ p+r & q+s \end{pmatrix}.$$

(b) Setting $AB = I_2$ gives $2p + r = 1$, $p + r = 0$, $2q + s = 0$, and $q + s = 1$. Subtracting within each pair gives $p = 1$ and $q = -1$. Hence $r = -1$ and $s = 2$.

(c)
$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2-1 & 1-1 \\ -2+2 & -1+2 \end{pmatrix} = I_2.$$

Thus $A^{-1} = B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

(d) Multiplying the equation on the left by A^{-1} gives
$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5-3 \\ -5+6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The solution is unique because multiplication by A^{-1} forces this value, and $AA^{-1} = I_2$ ensures it satisfies the original equation.