Additional Practice: Linear Algebra and Matrix Algebra

MA180/MA185/MA190

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This sheet provides additional practice problems. This is not marked, nor is it part of the tutorials. Although the solutions are available, try to get to the point where you can write a complete answer without looking at them.

1 Linear systems and modelling

1. Consider the linear system

$$\begin{cases} x - 2y + z = 1, \\ 2x + y - 3z = -4, \\ -x + 4y + 2z = 9. \end{cases}$$

   (a) Write down its augmented matrix.

   (b) Write the system in the form $Av = b$, giving A, v, and b.

   (c) State the dimensions of A, v, and b.

2. A parabola has the equation $y = ax^2 + bx + c$ and passes through the points $(0, 2)$, $(1, 1)$, and $(-1, 5)$.

   (a) Form a system of three linear equations for a, b, c.

   (b) Solve the system and give the equation of the parabola.

   (c) Verify that your equation passes through all three points.

3. A meal is made from three foods, X, Y, and Z. Each 100-gram portion provides the following

units of fat, protein, and carbohydrates:

	fat	protein	carbohydrates
X	2	1	3
Y	1	2	2
Z	3	1	4

A meal must provide 14 units of fat, 7 of protein, and 20 of carbohydrates.

(a) Let x, y, z be the numbers of 100-gram portions of the three foods. Form the corresponding linear system.

(b) Write it as $Av = b$ and solve it by Gaussian elimination.

(c) Interpret your solution in terms of portions of food.

2 Row operations and Gaussian elimination

4. Starting with the augmented matrix

$$\left(\begin{array}{cc|c} 0 & 2 & 4 \\ 1 & 1 & 3 \end{array}\right),$$

perform these row operations, writing the matrix after *each* step:

$$R_1 \leftrightarrow R_2, \qquad R_2 \longrightarrow \tfrac{1}{2}R_2, \qquad R_1 \longrightarrow R_1 - R_2.$$

Identify which of the three fundamental types of row operation is used at each step. Read off the solution of the original system.

5. Solve the following system using Gaussian elimination. Display an augmented matrix in row echelon form (REF), and use back substitution to obtain the solution.

$$\begin{cases} x + 2y - z = 0, \\ 2x + 5y + z = 10, \\ -x + y + 2z = 6. \end{cases}$$

Check your answer in the original equations.

6. Solve the system

$$\begin{cases} 2y + z = 0, \\ x - y + 2z = 6, \\ 2x + y + z = 3. \end{cases}$$

by row reduction. Explain why interchanging two rows is a useful first step. Continue to REF and solve by back substitution.

7. Row-reduce the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

to reduced row echelon form (RREF).

(a) Write down the resulting equations.

(b) Describe *all* solutions (x, y, z), using one variable of your choice to express the others.

(c) Give two different solutions and explain why there are infinitely many.

8. For each matrix below, decide whether it is in REF, in RREF, or in neither form. Identify its pivots and briefly justify your answer.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

3 Number of solutions of linear systems

9. Each matrix below is an augmented matrix for a linear system. Determine whether the corresponding system has a unique solution, no solution, or infinitely many solutions. Explain your conclusion using the rows of the matrix.

(a) $\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 2 & 4 \end{array}\right)$;

(b) $\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 0 & 3 \end{array}\right)$;

(c) $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

10. Consider the system, where t and b are real constants,

$$\begin{cases} x + y = 3, \\ 2x + ty = b. \end{cases}$$

Use row operations to determine the number of solutions in each case:

(a) $t \neq 2$;

(b) $t = 2$ and $b = 6$;

(c) $t = 2$ and $b \neq 6$.

In the first case, find the unique solution in terms of t and b.

11. *Challenge.* Let $t, s \in \mathbb{R}$. Consider

$$\begin{cases} x + y + z = 3, \\ x + 2y + 3z = 7, \\ 2x + 3y + (4 + t)z = 10 + s. \end{cases}$$

Subtract a suitable combination of the first two rows from the third. Determine exactly which values of t and s give

(a) a unique solution;

(b) no solution;

(c) infinitely many solutions.

Justify all three cases without using determinants.

4 Matrix dimensions, entries, and arithmetic

12. Let
$$M = \begin{pmatrix} 0 & -2 & \pi & 4 \\ 3 & 1 & -1 & 5 \end{pmatrix}.$$
(a) State the dimensions of M.
(b) Find M_{14}, M_{21}, M_{23}, and M_{24}.
(c) What are the dimensions of $-3M$?
(d) If N is a 4×2 matrix and P is a 2×4 matrix, which of $M + N$ and $M + P$ is defined? Explain.

13. For
$$P = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 4 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 5 & 2 \\ -1 & 3 & 1 \end{pmatrix},$$
calculate
(a) $P + Q$;
(b) $P - Q$;
(c) $-2P + 3Q$.
Show the entry-wise arithmetic in at least one part.

14. Suppose X is a 2×2 matrix satisfying
$$2X + A = 3B, \qquad A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 3 \\ 4 & -1 \end{pmatrix}.$$
Find X and verify your answer by substituting it into the matrix equation. Why must X have the same dimensions as A and B?

5 Matrix multiplication and column vectors

15. Suppose A, B, C, and D have dimensions
$$A : 2 \times 3, \qquad B : 3 \times 2, \qquad C : 2 \times 2, \qquad D : 3 \times 1.$$
For each of AB, BA, AC, CA, BC, CD, and DA, state whether the product is defined. If it is, give the dimensions of the product. No numerical entries are needed.

16. Let
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}.$$
(a) Calculate AB and BA by row-column multiplication.
(b) Are the two products equal?
(c) Explain what this example shows about commutativity of matrix multiplication.

17. Consider the matrix equation $Av = b$, where
$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \qquad v = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad b = \begin{pmatrix} 8 \\ 1 \end{pmatrix}.$$

(a) Write out the corresponding system of linear equations.

(b) Solve it using an augmented matrix and row operations.

(c) Verify the result by calculating Av directly.

6 Matrix identities and special matrices

18. Decide whether each statement is true or false. Give a brief explanation; when a statement is false, give a counterexample using matrices of suitable dimensions.

(a) $A + B = B + A$, whenever $A + B$ is defined.

(b) $AB = BA$, whenever both AB and BA are defined.

(c) $A(B + C) = AB + AC$, whenever the indicated operations are defined.

(d) If AB is defined, then BA must also be defined.

(e) $A + Z = A$ when Z is the zero matrix of the same dimensions as A.

(f) Every diagonal matrix is an identity matrix.

In your explanations, distinguish between the zero, diagonal, and identity matrices where relevant.

7 Determinants and inverses of 2×2 matrices

19. Consider

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}, \qquad C_t = \begin{pmatrix} t & 2 \\ 3 & t-1 \end{pmatrix}.$$

(a) Calculate $\det(A)$ and $\det(B)$.

(b) Decide which of A and B is invertible, giving a reason.

(c) Find $\det(C_t)$ and determine *all* real values of t for which C_t is singular.

(d) For which values of t is C_t invertible?

20. *Challenge.* Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

(a) Calculate AB in terms of p, q, r, s.

(b) Set $AB = I_2$ and solve the resulting equations for p, q, r, s.

(c) Check that your matrix B also satisfies $BA = I_2$. What is A^{-1}?

(d) Use A^{-1} to solve

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Explain why the solution is unique.