To find an inverse linear transform. it suffices to find the matrix inverse of the corresponding matrix.

Studying properties of matrices

Let A be ~~matrix~~ $n \times n$ matrix.

Def. A non-zero vector $v \in \mathbb{R}^n$ is an eigenvector of A if there is a $\lambda \in \mathbb{R}$ such that

$$Av = \lambda \cdot v$$

(λ: eigenvalue.)

Ex: $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Av = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

From this, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 4.

$v = \underline{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}$.

$$Av = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} = 4 \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$Av = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 7/6 \end{pmatrix}$$

This v is <u>not</u> an eigenvector!

$v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$Av = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

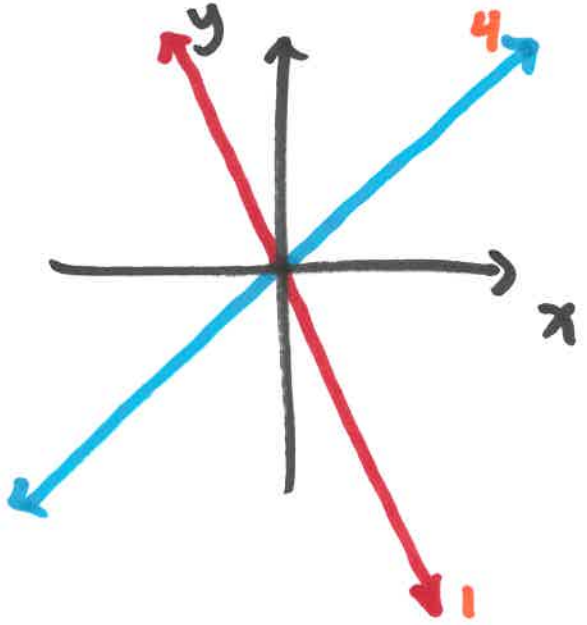
From this, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 1.

Fact: If v is an e-vect. of A, then so is $k \cdot v$ for all non-zero k.

$$Av = \lambda v$$

$$A(kv) = k(Av) = k\lambda v = \lambda(kv)$$

Once you have an eigenvector — you get a line of eigenvectors.



$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

E-vects and e-vals are have immense application.

- Principal component analysis / singular value decomposition

- Google: developed "pagerank" which uses the mathematics of e-vects & e-vals.

How to hunt for eigenvectors?

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$(A - \lambda I)v = 0$$

An eigenvector multiplies to 0 under

$$A - \lambda I$$

equivalently: $\lambda I - A$. Slightly more convenient.

This implies that

$$\det(A - \lambda I) = 0.$$

Use this → to solve for eigenvalues!

Ex: $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. Find its eigenvectors, if they exist.

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

variable λ

$$0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

(A, λI_2)

$$= \det\left(\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix}\right)$$

$$= (1-\lambda)(2-\lambda) - 12$$

$$= \lambda^2 - 3\lambda - 10$$

$$= (\lambda - 5)(\lambda + 2)$$

The e-vals are $\lambda = -2$ and $\lambda = 5$.

We'll find the e-vect for $\lambda = -2$.

$$A - \lambda I = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}$$

Need $v \in \mathbb{R}^2$ such that

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$3x + 3y = 0$

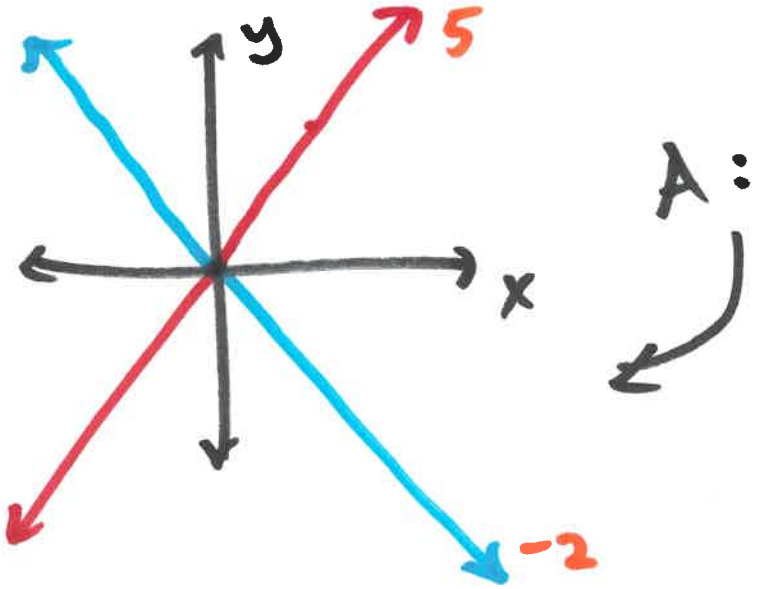
$4x + 4y = 0$

An eigenvector is $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\lambda = 5$:

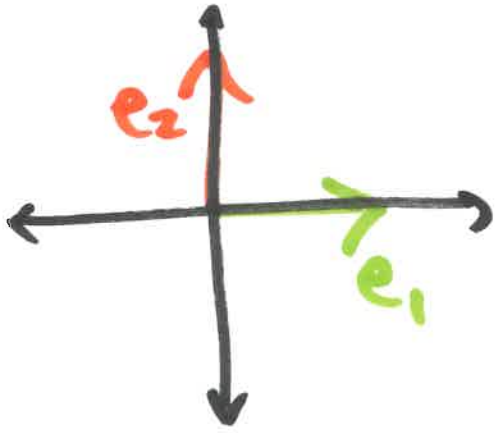
$$A - \lambda I = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix}$$

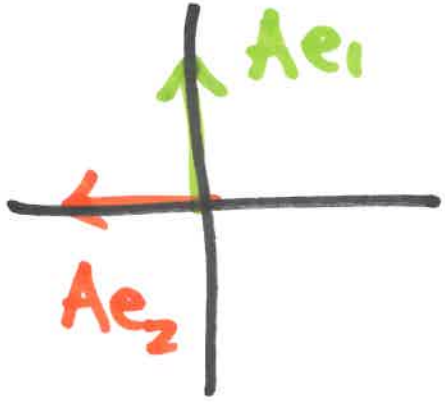
An eigenvector is $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$



A:

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.





You won't find two lines like previous two examples.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$$

No real roots: $i, -i$ $\longrightarrow$ $\lambda^2 + 1 = 0$.