Linear transforms and matrices are one in the same.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$$

An element of $\mathbb{R}^n$ is a tuple with n real numbers. For us in the linear alg. context, they are col. vectors (i.e. matrices of dim $n \times 1$).

$\mathbb{R}^2$ — Euclidean plane / xy-plane

Ex: $A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. Find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[-2\cdot R_1 \rightsquigarrow R_3]{-2\cdot R_1 \rightsquigarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -2 & 1 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{-1\cdot R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 & 2 & -1 \\ 0 & -2 & 4 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[2\cdot R_2 \rightsquigarrow R_3]{-1\cdot R_2 \rightsquigarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 & 2 & -2 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & 1 & -1 \end{array}\right)$$

$-1 \cdot R_3 \curvearrowright R_1$

$R_3 \curvearrowright R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -2 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -1 \end{array}\right)$$

I $\qquad$ A^{-1}

$AA^{-1} = I = A^{-1}A.$

We see that

$AA^{-1} = \begin{pmatrix} & & 2 \\ & & \\ & & \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & -2 & 2 \\ \frac{1}{2} & 3 & -2 \\ \frac{1}{2} & 1 & -1 \end{pmatrix}$$

Ex: Find a solution to:

$$\begin{aligned} 2x + 4z &= 5 \\ x + y &= -2 \\ 2x + y + z &= -1 \end{aligned}$$

$$\begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

This is $Av = b$, where A is from the previous example.

$$A^{-1}Av = A^{-1}b$$

$$Iv = A^{-1}b$$

$$v = A^{-1}b$$

$$v = \begin{pmatrix} -1/2 & -2 & 2 \\ 1/2 & 3 & -2 \\ 1/2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - 5/2 \\ -4 + 5/2 \\ -1 + 5/2 \end{pmatrix}$$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x + 3y + 2z \\ 3x + 3y + 2z \\ y + z \end{pmatrix}.$$

Find the matrix A relative to the standard basis such that $T = T_A$.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Ex: What is the image of the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ under T?

In other words: what is $T\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$?

$$T\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ 1 \end{pmatrix}$$

Ex: Does there exist a vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that

$$T(v) = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}?$$

In other words: what vector v has image $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ under T?

We need to find v such that

$$Av = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}, \quad \text{with } A = \begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

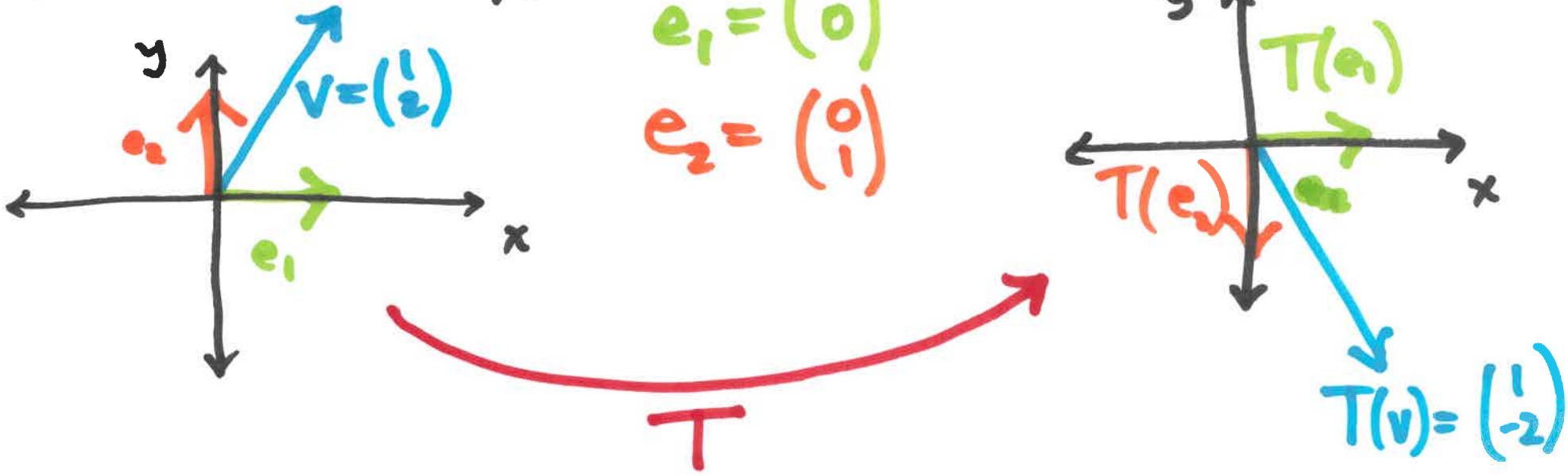
Thus, we need to solve the corresponding linear system. Try yourself!

Geometry with L.T.s

Look primarily at $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Fact: Reflections about lines in $\mathbb{R}^2$ are linear transformations.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection about the x-axis. Find A such that $T = T_A$.



$$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad v$$

match!