Whenever $\det(A) \neq 0$, we have

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$ $\det(A) = 1 = 2\cdot(-1) - (1)(-3)$

$$A^{-1} = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Algorithm for computing inverses.

Assume A is square. Form augmented Matrix

identity

$$(A \mid I) \xrightarrow{\text{Gaussian elim.}} (I \mid A^{-1})$$

# of rows of A = # of rows of I

We get A into RREF and the I comes along for the ride. We do not use any pivots in I.

This only works when the RREF of A is I. Otherwise, no inverse!

Linear transformations.

function f assigns output values to elements of a set X. We write $f: X \to Y$

$$\boxed{y = f(x)}$$

for $x \in X$ and $y \in Y$. We say y is the image of x (under f). The set of all images (of all elements in X) is called the range of function. Here, X is the domain, and Y is the codomain.

In our context X and Y are vector spaces, i.e., $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.

Functions between vector spaces are typically called **transformations**, typically written

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

One kind of transformation is given by a matrix A:

$$T_A(\underline{x}) = \underline{A}\underline{x}$$

$n \times 1$ $\quad$ $m \times n$ $n \times 1$

Ex: $A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}$, $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Let's see what T_A "looks like".

$$T_A\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 3 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x - y \\ -x \end{pmatrix}$$

So the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ under T_A is

$$T_A\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 7 \\ -5 \end{pmatrix}$$

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

- for all $k \in \mathbb{R}$, for all $x \in \mathbb{R}^n$, $T(k \cdot x) = k \cdot T(x)$.
- for all $x, y \in \mathbb{R}^n$, $T(x+y) = T(x) + T(y)$.

Fact: T_A is linear.

Good sanity check: take $k = 0$. Linear implies

$T(\underline{0}) = \underline{0}$ (underline to emphasize zero vector.)

All linear transformations are of the form T_A for some matrix A.

Standard basis

The standard basis of $\mathbb{R}^n$ is a set of n vectors written

$$\{e_1, e_2, \ldots, e_n\}.$$

The vector e_i (for $i=1,2,\ldots,n$) has a 1 in the ith position and zeros elsewhere

Ex: $n=3$

$$e_1=\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad e_2=\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

columns of I!

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a Linear trans.

$$A = \begin{pmatrix} | & | & | & & | \\ T(e_1) & T(e_2) & T(e_3) & \cdots & T(e_n) \\ | & | & | & & | \end{pmatrix}$$

The claim is that

$x \in \mathbb{R}^n \qquad T(x) = Ax = T_A(x)$

(This follows from linearity of T.)

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-3y \\ -x+y \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{pmatrix}$$

$$T = T_A : \begin{pmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-3y \\ -x+y \end{pmatrix}$$