Graph Theory

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IMO Problems

IMO 1964 (Moscow)

Problem 4. Seventeen people correspond by mail with one another—each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

IMO 1991 (Sigtuna)

Problem 4. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

IMO 2005 (Mérida)

Problem 6. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

IMO 2007 (Hanoi)

Problem 3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a **clique** if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its **size**.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

IMO 2022 (Oslo)

Problem 6. Let n be a positive integer. A **Nordic square** is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing larger numbers is called a **valley**. An **uphill path** is a sequence of one or more cells such that: 1. the first cell in the sequence is a valley, 2. each subseequent cell in the sequence is adjacent to the previous cell, and 3. the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of upill paths in a Nordic square.

Basic definitions

A **(simple) graph** G is a pair of sets V and E, called vertices and edges respectively, with the property that every edge $e \in E$ is a subset of V of cardinality two. We write $V(G)$ and $E(G)$ to signify the vertex set and edge set of G.

A **directed graph** G is a simple graph where each $e \in E$ has an orientation: a start and end. More precisely, each $e \in E$ is an element of $V \times V$. By default we assume our graphs are undirected.

Two vertices u and v of a graph G are **adjacent**, denoted by $u \sim v$, if $\{u, v\}$ is an edge of G. An edge $\{u, v\}$ is **incident** to both u and v. Vertices x and y of G are **connected** if there exists a (finite) sequence of vertices in G of the form

$$x = v_0 \sim v_1 \sim \cdots \sim v_{n-1} \sim v_n = y.$$

We call the sequence of vertices $(v_0, \ldots, v_n)$ above a **path**, and if $v_0 = v_n$, then the path is called a **cycle**. The **length** of a path (and also a cycle) is equal to the number of edges traversed; the sequence above between x and y has lengh n.

A **subgraph** H of a graph G is a graph such that $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$.

An **induced subgraph** I of a graph G is a subgraph of G such that $E(I)$ contains all edges between the vertices $V(I)$.

A graph G is **connected** if every pair of vertices of G are connected. The **connected components** of G are the maximal connceted subgraphs of G.

The **degree** of a vertex v in a graph G is equal to the number of vertices adjacent to v, written $\deg v$.

The **distance** between vertices u and v of a connected graph G is equal to the minimal length of all the paths between u and v in G.

A subset U of vertices V in a graph forms a **clique** if every pair of vertices in U are adjacent.

Two graphs G and H are **isomorphic** if there is a bijection $f : V(G) \to V(H)$ with the property that $u \sim v$ in G if and only if $f(u) \sim f(v)$ in H.

Families of graphs

You should try draw many different examples of these graphs yourself.

Empty graphs: Graphs with no edges.

Complete graphs: Graphs where every pair of vertices are adjacent. The complete graph on n vertices is called K_n.

Cycle & path graphs: Graphs of an individual cycle and path, respectively. The cycle and path graphs on n vertices are called C_n and P_n, respectively.

Trees: Graphs with no cycles.

Bipartite graphs: Graphs where the vertices V can be partitioned into two subsets U_1 and U_2 such that every edge is incident to one vertex from U_1 and one vertex from U_2.

Complete bipartite graphs: Bipartite graphs such that every vertex in U_1 is adjacent to every vertex in U_2. The complete bipartite graph with $\#U_1 = m$ and $\#U_2 = n$ is $K_{m,n}$.

Planar graphs: Graphs that can be drawn in the plane (e.g. on a piece of paper) where edges only intersect at vertices.

Problems

1. What is the number of edges in K_n? What about $K_{m,n}$?
2. Let G be a connected graph. Prove that two paths which are both a longest path in G contain at least one vertex in common.
3. Let G be a graph with n vertices $v_1, \ldots, v_n$ and m edges. Suppose that $d_1, \ldots, d_n$ are the degrees of the vertices $v_1, \ldots v_n$. Show that
$$\sum_{i=1}^{n} d_i = 2m.$$
4. For a graph G, let $\Delta(G)$ be the maximal degree of its vertices. Characterise all graphs with $\Delta(G) \leqslant 2$. Characterise all graphs with $\Delta(G) = 2$.
5. Show that K_5 and $K_{3,3}$ are not planar graphs.
6. Prove that a graph is bipartite if and only if it does not contain a cycle of odd length.

7. For a graph G, define the **complement graph** $\overline{G}$ such that $V(\overline{G}) = V(G)$ and $E(\overline{G})$ comprises all the edges $\{u, v\}$ with u and v not adjacent in G. Show that if G is not connected, then $\overline{G}$ is connected.
8. For which $n \geqslant 3$ is the cycle graph C_n bipartite? For $n \geqslant 2$ is the path graph P_n bipartite?
9. For each of the following, construct two non-isomorphic graphs satisfying the properties, or explain why it is impossible.
   (a) A tree with 6 vertices with vertex degrees $(3, 2, 2, 1, 1, 1)$.
   (b) A tree with 5 vertices with vertex degrees $(3, 2, 1, 1, 1)$.
   (c) A graph with 8 vertices all of degree 2.
   (d) A graph with 5 vertices all of degree 3.
10. The following graphs are all isomorphic and are called the Petersen graph (the first one being the typical one). Construct isomorphisms between all the graphs.