Submission to Assignment 1

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1 Free and open source software

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2 Enumeration in finite vector spaces

We assume some background on finite fields; see for example [1, Section 14.3]. We aim to prove that the number of k-dimensional subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$ (where q is a prime power) is given by the q-binomial coefficient $\binom{n}{k}_q$. Recall, for $0 \leqslant k \leqslant n$ the q-binomial coefficient is defined as

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

Theorem 2.1. *The number of k-dimensional subspaces of an n-dimensional vector space V over $\mathbb{F}_q$ is $\binom{n}{k}_q$.*

Proof. We first note that a k-dimensional subspace of V has a basis consisting of k vectors. There are q^n-1 choices for the first vector (every non-zero vector in V). For the second vector, we need to pick a vector not in the span of the first, leaving us $q^n - q$ choices. This process continues, reducing the number of choices by a factor of q each time until we select the k-th vector. Thus, the total number of ways to choose a sequence of k linearly independent vectors in V is

$$(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{k-1}).$$

However, each k-dimensional subspace has exactly $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$ such sequences—corresponding to the different bases of the subspace, so we divide by this number to count each subspace once. Thus, the number of k-dimensional subspaces is

$$\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \binom{n}{k}_q.$$

□

3 The Fano plane

The *Fano plane* is the smallest finite projective plane. We can associate each point of the Fano plane with a 1-dimensional subspace from $\mathbb{F}_2^3$, and each line with a 2-dimensional subspace. An illustration of the Fano plane is given in Figure 1. By Theorem 2.1, the number of points is equal to

$$\binom{3}{1}_2 = \frac{(2^3-1)}{(2^1-1)} = 7,$$

and the number of lines is equal to

$$\begin{aligned}\binom{3}{2}_2 &= \frac{(2^3-1)(2^3-2^1)}{(2^2-1)(2^2-2)} \\ &= \frac{7\cdot 6}{3\cdot 2} = 7.\end{aligned}$$

Therefore, there are 7 points and 7 lines.

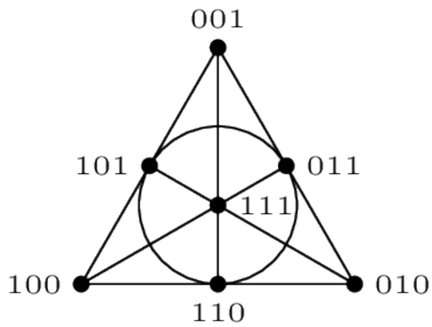

Figure 1: A representation of the Fano plane.

References

[1] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.