Implications of Lyapunov functions

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Prop. 9.9 If the system $x' = f(x)$ has a strict Lyapunov function in a neighborhood Ω around x_0, then x_0 is asymptotically stable.

proof: Assume $x(t)$ is a sol'n and L a strict. L.f. Let $\varepsilon > 0$, and take δ to be the same δ as in Prop. 9.8. That is, for $\|x(0) - x_0\| < \delta$, then $\|x(t) - x_0\| \leq \varepsilon/2$ for all $t \geq 0$. Since $L(x)$ is decreasing and bounded below, the limit

$$\lim_{t\to\infty} L(x(t)) =: L_0.$$

exists.

Goal: Show $L_0 = 0$. If true, then by definition

$$\lim_{t\to\infty} L(x(t)) = 0 = L(x_0) \Longrightarrow \lim_{t\to\infty} x(t) = x_0.$$

by continuity. asymptotically stable.

Assume $L_0 \neq 0$.

By the continuity of L $\exists r > 0$ s.t. $r < \varepsilon/2$ and $L(x) = \|L(x)\| < L_0$ whenever $\|x - x_0\| < r$.

Let

$$A = \{x \in \mathbb{R}^d : r \leq \|x - x_0\| \leq \varepsilon/2\}$$

$-\varepsilon$:

$$\Delta = \{x \in \mathbb{R}^d : r \leq \|x - x_0\| \leq \varepsilon/2\}.$$

Since Δ is closed & bounded and L_f' is continuous, the number

$$k = \max_{x \in \Delta} \{L_f'(x)\}$$

exists. Because L is strict, $k < 0$.

For $\|x(0) - x_0\| \leq \delta$, we know $\|x(t) - x_0\| \leq \varepsilon/2$.

Thus, $x(t) \in \Delta$ for all $t \geq 0$. Thus,

$$L_f'(x(t)) \leq k < 0$$

for all $t \geq 0$. Thus, this implies that

$$\lim_{t \to \infty} L(x(t)) = -\infty.$$

This is a contradiction! L_0 exists! Thus, $L_0 = 0$.

$\blacksquare$