Stability via Lyapunov functions

Thursday, July 9, 2020 8:27 PM

Ex 9.11:

$$x' = -2x - y$$
$$y' = -x^2 - 4x - y.$$

Here $f = (-2x - y, -x^2 - 4x - y)$. The fixed points are $(0,0)$, $(-2,4)$.

The Lyapunov function for $(-2,4)$ was

$$L(x,y) = (x+2)^2 + (y-4)^2.$$

This is strict!

$$L_f'(x,y) < 0$$

for $(x,y) \in \Omega \setminus \{(-2,4)\}$. By Prop 9.9, $(-2,4)$ is asymp. stable.

Consider $(0,0)$. For this point no Lyapunov functions exist!

$$Df = \begin{pmatrix} -2 & -1 \\ -2x-4 & -1 \end{pmatrix} \xrightarrow{@(0,0)} \begin{pmatrix} -2 & -1 \\ -4 & -1 \end{pmatrix}$$

$$\det \begin{vmatrix} \lambda+2 & 1 \\ 4 & \lambda+1 \end{vmatrix} = (\lambda+2)(\lambda+1) - 4$$
$$= \lambda^2 + 3\lambda - 2$$

$$\lambda = \frac{-3 \pm \sqrt{9-4(-2)}}{2}$$

$$\lambda = \frac{-3 \pm \sqrt{17}}{2} \qquad \sqrt{17} \approx 4.$$

The Df has one positive and one negative eigenvalue. $\blacksquare$

Common trick: define a Lyapunov function

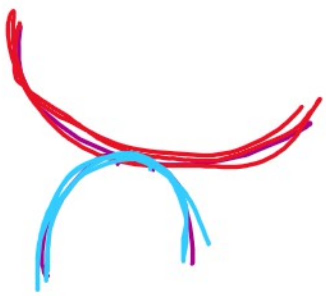
$$L(x,y) = A(x-x_0)^{\alpha} + B(y-y_0)^{\beta},$$

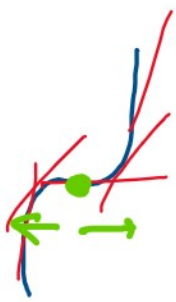
where α, β are even and $A, B > 0$.
This correspond (x_0, y_0).

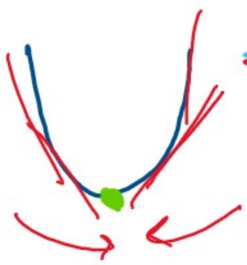
(x_0, y_0)

$L(x,y) > 0.$

$L_f'(x,y) < 0$







Ex 9.12:

$$x' = -x^3 + y^2$$
$$y' = -2xy - y.$$

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Only one fixed pt. $(0,0)$.

$$Df(x,y) = \begin{pmatrix} -3x^2 & 2y \\ -2y & -2x-1 \end{pmatrix} \xrightarrow{@(0,0)} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Jacobian method inconclusive.

Lyapunov. $L(x,y) = \underline{A}x^2 + \underline{B}y^2$.

Because $A, B > 0$, then we satisfy:

1. $L(0,0) = 0$

2. $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $L(x,y) > 0$.

$$L_f'(x,y) = (2Ax, 2By) \cdot (-x^3 + y^2, -2xy - y)$$

$$= \underline{-2Ax^4} + \underbrace{\underline{2Axy^2} - \underline{4Bxy^2}}_{(2A-4B)xy^2} \underline{-2By^2}.$$

Want $2A - 4B = 0$. Choose $A = 2$, $B = 1$.

$$L(x,y) = 2x^2 + y^2.$$

This is a Lyapunov function. In fact, it strict!. By Prop 9.9, the fixed pt. $(0,0)$ is asymptotically stable pt $(0,0)$ is asymptotically stable. $\blacksquare$