Properties of matrix exponentials

Sunday, June 28, 2020 9:10 PM

Prop 8.3: Suppose $A, B, S \in Mat_{d\times d}(\mathbb{R})$, where S is invertible. If A and B commute ($AB = BA$),

1. $e^{A+B} = e^A e^B$,
2. $(e^A)^{-1} = e^{-A}$,
3. $(e^A)^m = e^{mA}$, $m \in \mathbb{Z}$,
4. $e^{S^{-1}AS} = S S^{-1} e^A S S^{-1}$

Ex 8.7: Let

$$A = \begin{pmatrix} -4 & 6 & -3 \\ 0 & 2 & 0 \\ 6 & -6 & 5 \end{pmatrix}.$$

The eigenvalues of A are -1, 2, and 2 with evecs

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Def: A matrix $A \in Mat_{d\times d}(\mathbb{R})$ is **diagonalizable** if there exists an invertible matrix S and a diagonal invertible matrix S and a diagonal matrix D s.t.

$$S^{-1}AS = D.$$

Since 3 evecs, A is diagonalizable.

$$S = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & & 0 \\ & 2 & \\ 0 & & 2 \end{pmatrix},$$

we know $S^{-1}AS = D$.

From Prop.

$$e^{S^{-1}AS} = S^{-1}e^{A}S$$

$$Se^{D}S^{-1} = \underbrace{Se^{S^{-1}AS}S^{-1}} = e^{A}$$

$$Se^{D}S^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} e^{-1} & & \cdot \\ & e^{2} & \\ \cdot & & e^{2} \end{pmatrix} \begin{pmatrix} -2 & 2 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-1}-e^{2} & -2e^{-1}+2e^{2} & e^{-1}-e^{2} \\ 0 & e^{2} & 0 \\ -2e^{-1}+2e^{2} & 2e^{-1}-2e^{2} & -e^{-1}+2e^{2} \end{pmatrix}. \quad \square$$

Def. A matrix $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ is nilpotent if $\exists n \in \mathbb{N}$ s.t. A^{n} is the zero matrix if $\exists n \in \mathbb{N}$ s.t. A^n is the zero matrix

If A is nilpotent, say, $A^n = 0$, then $A^k = 0$ for all $k \geq n$.

$$A^k = \boxed{A^n} \cdot A^{k-n} = 0 \cdot A^{k-n} = 0.$$

$$e^A = \underbrace{\sum_{m=0}^{\infty} \frac{A^m}{m!}}_{\text{infinite}} = \underbrace{\sum_{m=0}^{n-1} \frac{A^m}{m!}}_{\text{finite}} + 0.$$

Ex 8.9: Let

$$A = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $A^3 = 0$, $\quad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

$$e^A = \sum_{m=0}^{2} \frac{A^m}{m!} = I + A + \frac{A^2}{2}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 4 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1/2 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 9/2 \\ & 1 & 1 \\ & & 1 \end{pmatrix}. \qquad \square$$

Question: What is the exponential of

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I + \underline{B}$$