Faculty for Mathematics
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Summer Semester 2020

Problem Set 9

Each problem is worth 2 points. The set is due on Wednesday 24 June by 23:59.

1. Consider the dynamical system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sin(x_n) + y_n \\ \frac{1}{2}y_n + x_n^2 \end{pmatrix}$$

   Prove that $(0, 0)$ is an attracting fixed point of the system.

2. Consider the discrete dynamical system given by the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$F(x, y) = \left(\frac{1}{2}\left(x + x^3\right), \frac{2y}{1 + 2x^2} \right).$$

   (a) Determine all of the fixed points of the system.

   (b) Show that all fixed points are saddle points.

   (c) Sketch the phase portrait of F.

3. Let $f_a : \mathbb{R} \to \mathbb{R}$ be the logistic map given by $f_a(x) = ax(1 - x)$.

   (a) Show that $f_4([0, 1]) \subseteq [0, 1]$.

   (b) For $a > 0$ and $a \neq 2$, what is the Lyapunov exponent of the orbit of $x_0 = 1/a$ under f_a?

   (c) Why is $a = 2$ excluded in part (b)? What is the Lyapunov number of the orbit of $x_0 = 1/2$ when $a > 0$?

4. We continue investigating the logistic map as in Problem 3.

   (a) Show that the orbit of $x_0 = 1/2$ under f_4 is unstable with respect to $[0, 1]$.

   (b) Compare this with the case $a = 1$: what is the stability of the orbit of $x_0 = 1/2$ under f_1 with respect to $[0, 1]$?