Faculty for Mathematics
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Problem Set 8

Each problem is worth 2 points. The set is due on Wednesday 17 June by 23:59.

1. Let $\lambda \in \mathbb{R}$ and set

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

   (a) Prove that for integers $n \geq 1$

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

   (b) Under what condition(s) is J invertible? In this case, calculate J^{-1} and J^{-n}.

   (c) Show that if $|\lambda| > 1$, then the sequence

$$(\|J^n x\|)_{n=1}^{\infty} \tag{1}$$

   diverges to infinity for all $x \in \mathbb{R}^2 \setminus \{0\}$.

   (d) Show that if $|\lambda| < 1$, then the sequence in (1) converges to the zero vector for all $x \in \mathbb{R}^2$.

2. Consider the two dimensional linear dynamical system $x_{n+1} = Ax_n$ given by the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

   (a) Find the eigenvalues of A. Does the system have any fixed points other than $x = 0$?

   (b) By writing A in the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

   show that the trajectories of the system on $\mathbb{R}^2$ are rotations and, hence, determine the periodic points of the system.
   (*Hint: note the minus sign; also θ is a "standard" angle—nothing complicated.*)

   (c) Show that $x = 0$ is a stable but not an attracting fixed point of the system.

3. Consider the two dimensional linear dynamical system $x_{n+1} = Ax_n$ given by the matrix

$$A = \begin{pmatrix} 4 & -3 \\ 3/2 & -3/2 \end{pmatrix}.$$

   Find the eigenvalues and eigenvectors of A and, hence, the stable and unstable subspaces of the dynamical system.