Faculty for Mathematics
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Problem Set 7

Each problem is worth 2 points. The set is due on Wednesday 10 June by 23:59.

1. Consider the following family of dynamical systems

$$f_a(x) = ax + x^3.$$

   Discuss the bifurcation that occurs at $a = 1$, and sketch the corresponding diagram near $a = 1$.

2. As discussed in Example 4.12, let

$$f_a(x) = x - x(x^2 - a)(x^2 - 4a).$$

   Show that for $a = 0$, the fixed point $x = 0$ is attracting and stable.

3. As discussed in Example 4.11, let

$$f_a(x) = x - (x^2 - a)(x^2 - 4a).$$

   (a) Show that for $a = 0$, the fixed point $x = 0$ is neither attracting nor repelling and is hence unstable.

   (b) For which values of a does another bifurcation happen?

4. Let $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ with a complex eigenvalue $\lambda = \alpha + i\beta$, with $\beta \neq 0$, and corresponding eigenvector $v = x + iy$, where $x, y \in \mathbb{R}^2$.

   (a) Show $Ax = \alpha x - \beta y$ and $Ay = \alpha y + \beta x$, and prove that $\overline{v} = x - iy$ is an eigenvector of A with eigenvalue $\overline{\lambda}$.

   (b) Prove that x and y are linearly independent.

   (c) Let S be the matrix whose columns are the vectors x and y, so that $S = (x, y)$. Show

$$S^{-1}AS = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$