Faculty for Mathematics
Bielefeld University

Joshua Maglione
Summer Semester 2020

Problem Set 5

Each problem is worth 2 points. The set is due on Wednesday 27 May by 23:59.

1. Consider the dynamical systems defined by the following functions. In each case, show that $x = 0$ is a fixed point of the dynamical system and discuss its stability properties.
   (a) $f(x) = -x - x^3$
   (b) $f(x) = -x + x^3$
   (c) $f(x) = -x + x^2$
   (d) $f(x) = -x - x^2$

2. Let $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = -x^2 + 5x - 4$.
   (a) Find the unique fixed point, $\bar{x}$, of f.
   (b) Show that for all $x \in B_1(\bar{x})$, the inequality $|f(x)| \leq |x|$ holds, with equality if and only if $x = \bar{x}$.
   (c) Show that $x = \bar{x}$ is neither an attracting nor a repelling fixed point.
   (d) Explain the difference between this situation and that of Proposition 3.8.

3. Consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} \frac{1}{2}\left(x + \frac{1}{2^k}\right) & \text{for } \frac{1}{2^k} < x \leq \frac{1}{2^{k-1}}, k \in \mathbb{N}, \\ 0 & \text{for } x = 0. \end{cases}$$

   (a) Show that for all $k \in \mathbb{N}$ and $x \in \left(1/2^k, 1/2^{k-1}\right]$,

$$\lim_{n\to\infty} f^n(x) = \frac{1}{2^k}.$$

   (*Hint: write $f^n(x)$ in terms of x, k, and n.*)
   (b) Show that the fixed point $x = 0$ is stable.

4. Consider the dynamical system $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2|x| - 1$. Show that for every $m \in \mathbb{N}$, there exists a periodic point of f with minimal period m.