Faculty for Mathematics
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Problem Set 4

Each problem is worth 2 points. The set is due on Wednesday 20 May by 23:59.

1. Consider the dynamical system defined by $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} 1 & \text{for } 1/4 < x < 1, \\ 1/2 & \text{for } x \in \{1/4, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Sketch the graph and the phase portrait.
   (b) Find the fixed points of f and describe their properties (i.e. stable, attracting, unstable, repelling).
   (c) Find the periodic orbits of f and describe their properties (i.e. stable, attracting, unstable, repelling).

2. Consider the dynamical system defined by $x_{n+1} = ax_n + b$. Use induction to prove that for all $n \in \mathbb{N}$,

$$x_n = \begin{cases} x_0 + nb & \text{if } a = 1, \\ a^n \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} & \text{if } a \neq 1. \end{cases}$$

3. (a) Prove that $x = 0$ is an attracting fixed point of the dynamical system defined by $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \sin(x^2)$.
   (b) Consider the system defined by $g : \mathbb{R} \to \mathbb{R}$ via $g(x) = \cos x$. By plotting $y = g(x)$ and $y = x$ on the same set of axes, show that g has one fixed point, $\bar{x} \in (0, \pi/2)$. Without attempting to calulate the value of $\bar{x}$, prove that it is an attracting fixed point.
   (c) Show that for all $b \in (0, 1)$, $g(x) = \cos(bx)$ has an attracting fixed point in $(0, \pi/2)$. What can happen for $b > 1$?
   (*Hint: to prove existence, consider $h(x) = g(x) - x$ and apply the Intermediate Value Theorem.*)

4. Consider the discrete dynamical system defined by the function $f : \mathbb{R} \to \mathbb{R}$, for $a \in \mathbb{R}$,

$$f(x) = ax + x^2.$$

   Find the fixed points and describe how their properties depend on the value of a.