Faculty for Mathematics
Bielefeld University

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Problem Set 3

Each problem is worth 2 points. The set is due on Wednesday 13 May by 23:59.

1. For $a = 2$, consider the dynamical system defined by the difference equation

$$x_{n+1} = ax_n(1 - x_n).$$

   (a) Find and describe the stability of its fixed points.

   (b) Compare with the cases $1 < a < 2$ and $1 = a$. What are the differences and similarities?

2. Let $X \subseteq \mathbb{R}$ and define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{4}\left(x^2 + 3\right).$$

   (a) Set $X = [0, 3/2]$. Show that $f(X) \subseteq X$ and that $f : X \to X$ is a contraction.

   (b) Find the fixed point(s) of f in $[0, 3/2]$ and describe their properties.

   (c) Now set $X = \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is not a contraction. Does f have any fixed points in $\mathbb{R} \setminus [0, 3/2]$? If so, do they have the same properties as those of (b)? If not, why not?

3. Let $f : \mathbb{Z} \to \mathbb{Z}$. Prove that for all $x \in \mathbb{Z}$ exactly one of the following statements is true.

   (i) The orbit of x is either periodic or eventually periodic.

   (ii) The orbit of x diverges.

   Hint: This is Theorem 2.13 in the lecture notes. Consider the proof of Theorem 2.12.

4. Let $f : I \to I$ be a continuous and strictly increasing function on the closed and bounded interval $I \subset \mathbb{R}$ and consider the time discrete system given by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}_0$.

   (a) Prove that there are no eventually periodic points in I. (Recall, that these are *not* periodic points by definition.)

   (b) Prove that any orbit under f is either constant (that is, we have a fixed point) or strictly monotonous.

   (c) Prove that any non-constant orbit converges to a fixed point.

   (d) Does the fixed point from (c) have to be unique, or may different orbits converge to different fixed points?