Problem Set 10

Each problem is worth 2 points. The set is due on Wednesday 1 July by 23:59.

1. Consider the initial value problem $x' = tx$ with initial condition $x(0) = x_0$.

   (a) On which subsets of $\mathbb{R} \times \mathbb{R}$ does the function $f(t, x) = tx$ satisfy the Lipschitz condition on?

   (b) Solve the differential equation. What can be said about the existence and uniqueness of solution(s) for every $x_0 \in \mathbb{R}$?

2. Consider the initial value problem $x' = \sqrt{x}$ with initial condition $x(0) = x_0$.

   (a) On which subsets of $\mathbb{R} \times [0, \infty)$ does the function $f(t, x) = \sqrt{x}$ satisfy the Lipschitz condition on?

   (b) Solve the differential equation. What can be said about the existence and uniqueness of solution(s) for every $x_0 \in \mathbb{R}$?

3. Let A and B be commuting $n \times n$ matrices and S an invertible $n \times n$ matrix. Prove the following identities.

   (a) $e^{A+B} = e^A e^B$. (*Hint: apply the Cauchy product formula and binomial theorem.*)

   (b) $e^{SAS^{-1}} = Se^A S^{-1}$.

4. Let $A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}$.

   (a) Write down the characteristic equation of A, and show that A fulfils it.

   (b) Diagonalize A and hence find A^{12} and e^A.