Faculty for Mathematics
Bielefeld University

Joshua Maglione
Summer Semester 2020

Problem Set 1

Each problem is worth 2 points. The set is due on Wednesday 29 April by 23:59.

1. For $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$, consider the difference equation

$$x_{n+1} = -x_n.$$

Find an expression for x_n in terms of x_0. What are the fixed points? Are there periodic points? If so, what is their minimum period; and if not, why not?

2. Fix $\phi \in [0, 2\pi)$, let $x_0 = (\sin \phi)^2$, and set

$$x_{n+1} = 4x_n(1 - x_n).$$

Describe the sequence $(x_0, x_1, x_2, x_3, \dots)$ (i.e. the orbit of x_0) for each of the initial values.

(a) $\phi = \pi/7$.
(b) $\phi = 2\pi/7$.
(c) $\phi = \pi/5$.
(d) $\phi = \pi/12$.

Hint: Among a few other trigonometric identities, apply the square of the "double angle" identity

$$(\sin(2\phi))^2 = 4(\sin(\phi))^2(\cos(\phi))^2.$$

3. For $a \in \mathbb{R}$ with $a > 0$, let

$$x_{n+1} = ax_n(1 - x_n).$$

(a) Describe the behavior of the system for initial states $x_0 = 0$ and $x_0 = 1$.
(b) Assuming $0 < a < 1$, show that for $0 < x_0 < 1$, $(x_n)_{n=0}^{\infty}$ is monotonically decreasing and bounded below. Conclude that the sequence converges to 0.
(c) Suppose $a \geq 1$.
    (i) Show that if $x_0 < 0$, then for all $n \in \mathbb{N}$, $x_n < 0$. Conclude that the sequence is monotonically decreasing. Does it converge? If so, what does it converge to; and if not, why not?
    (ii) What happens if $x_0 > 1$?

4. Solve the differential equation

$$x'(t) = (x(t))^2$$

for the initial value $x(0) = x_0 > 0$ in the time interval $0 \leq t < 1/x_0$. Sketch the graph. What happens at $t = 1/x_0$? Does the same situation arise if x_0 is negative and $t \geq 0$?