Lecture Notes for Optimization and Dynamics

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September 13, 2020

Based on notes of Terauds (2015) & Nilsson (2012, 2013).

Contents

Chapter 1

Introduction to dynamical systems

The fundamental idea of the course is to use mathematics to make predictions about the future. We do this all the time in many different contexts. For example:

(Finance) Trajectory of interest rates, inflation rate, and GDP,

(Finance) Call options for stocks,

(Biology) Spread of infectious diseases,

(Biology) Population growth,

(Physics) Heat transfer,

(Physics) Positions and velocities of planets and stars.

These are examples of different kinds of systems.

Definition 1.1. A **system** is a set of measurable quantities, and a **dynamical system** is a system that changes over time.

In order to analyze systems, we build (mathematical) **models**. There is no such thing as a perfect model! But we can build some incredibly accurate models for some systems.

Definition 1.2. The **state** of a system is the set of values describing the system at that time.

If we know the state of a system at some time $t = t_0$, the model allows us to predict the state of the system at some future time $t = t_n$. Every dynamical system consists of two parts:

1. the **state space**: the set of all possible states of the system and
2. the **time evolution rule**: the rule (function) that describes how the states of the system change over time.

A time evolution rule may take various forms, but we often try to convert it to one that gives the state of the system at a general time t in terms of an **initial state**.

1.1 Discrete and continuous time dynamical systems

Dynamical systems are classified as either discrete or continuous based on the nature of how the system changes over time.

Definition 1.3. A **discrete (time) dynamical system** is a dynamical system that changes state in discrete time steps. For example at $t_0, t_1, t_2, \dots$.

Consider, for example, the balance of a savings account where interest is compounded monthly. Discrete systems can be described by difference equations or recurrence relations.

Definition 1.4. A **continuous (time) dynamical system** is a dynamical system that changes state continuously over time.

Consider, for example, the position of a swinging pendulum, or the value of a commodity. Such systems can be described by differential equations. We now consider some more detailed examples of dynamical systems.

1.1.1 Newton–Raphson method

The Newton–Raphson method is an iterative numerical method for finding real roots of differentiable functions. It was first developed by Newton in *Method of Fluxions* around 1671 but published after Raphson's version around 1690, which is simpler than Newton's.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with a real root $\bar{x}$. Given a point $x_n \in \mathbb{R}$, we denote by T_n the tangent line to the curve f at the point $(x_n, f(x_n))$. We define a new point $x_{n+1} \in \mathbb{R}$ to be the intersection of this tangent line with the x-axis.

Recall, the slope of the tangent line is $T_n(x)$ is equal to the slope of the curve f at $(x_n, f(x_n))$. With $T_n(x) = mx + c$, we have

$$m = f'(x_n), \qquad\qquad f(x_n) = mx_n + c,$$

which gives $c = f(x_n) - f'(x_n)x_n$. The equation of the tangent line is therefore

$$\begin{aligned} T_n(x) &= f'(x_n)x + f(x_n) - f'(x_n)x_n \\ &= f'(x_n)(x - x_n) + f(x_n). \end{aligned}$$

This line intersects the x-axis when $T_n(x) = 0$. This occurs when

$$x = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Thus, we set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.1}$$

and we iterate this process to numerically approximate the root of f "near" some initial guess.

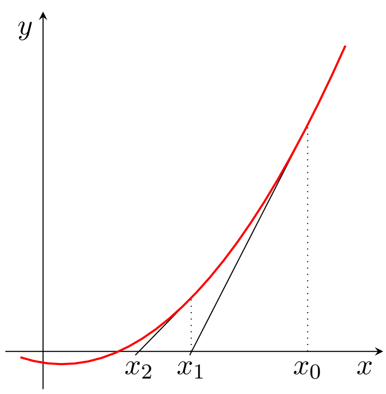


Figure 1.1: Illustrating the first few iterations of the Newton–Raphson method.

So how is this a dynamical system? The quantity that we are describing is the approximate value of the root. The state of the system at time (step) n is x_n, which may be any real number.

So the state space of the system is $\mathbb{R}$. The dynamical system is discrete since there is one step after another, and the time evolution rule is given by equation (1.1).

We start the iteration with an initial value x_0, the initial state of the system. This value *determines* whether or not the state converges to the actual root of the function. There are continuous functions for which no starting value will converge; see the exercises.

1.1.2 Exponential growth and decay

Sometimes we may choose whether to model a dynamical system with discrete or continuous time. In the next two examples, we compare discrete and continuous models of the same system concerning exponential growth/decay. This system is one of the most basic and fundamental systems.

First, we consider the discrete system. Let $a \in \mathbb{R}^+$, $x_0 \in \mathbb{R}$, and for all $n \in \mathbb{N}$,

$$x_{n+1} = ax_n, \tag{1.2}$$

By iterating, we get

$$\begin{aligned}
x_1 &= ax_0 \\
x_2 &= ax_1 = a^2 x_0 \\
x_3 &= ax_2 = a^3 x_0 \\
&\vdots \\
x_n &= ax_{n-1} = a^n x_0.
\end{aligned}$$

Now we explore the qualitative behavior of this system for different values of the constant a.

Case 1: $a = 1$. Then for all $n \in \mathbb{N}$, $x_n = x_0$, so the system is constant. This is uninteresting.

Case 2: $a > 1$. We consider three different cases based on the initial value x_0.

- If $x_0 = 0$, then $x_n = 0$ for all $n \in \mathbb{N}$. We say that $x = 0$ is a **fixed point**.
- If $x_0 > 0$, then $x_n > x_{n-1}$ for all n. That is, the sequence $(x_n)_{n\in\mathbb{N}}$ is monotone increasing and $x_n \to \infty$ as $n \to \infty$.
- If $x_0 < 0$, then $x_n < x_{n-1}$ for all n. That is, the sequence $(x_n)_{n\in\mathbb{N}}$ is monotone decreasing and $x_n \to -\infty$ as $n \to \infty$.

We will discuss this in more detail later, but for now we call $x = 0$ an **unstable fixed point** since the system is "going away" from $x = 0$. If the system has the state $x = 0$, then it will stay there. If we start the system just a small distance away from $x = 0$, then the value moves further away from this point.

Case 3: $0 < a < 1$. As in the case above, we consider three different situations.

- If $x_0 = 0$, then $x_n = 0$ for all $n \in \mathbb{N}$.
- If $x_0 > 0$, then $0 < x_n < x_{n-1}$ for all n, and $x_n \to 0$ as $n \to \infty$.
- If $x_0 < 0$, then $0 > x_n > x_{n-1}$ for all n, and $x_n \to 0$ as $n \to \infty$.

In this case, we say that $x = 0$ is a **stable fixed point** since the system is "coming towards" $x = 0$. We can perturb the starting point away from the state $x = 0$, and the system will evolve back to the state $x = 0$.

Remark 1.5. The equation $x_{n+1} = ax_n$ is called a **difference equation** as we can write it in the form of a difference:

$$\underbrace{x_{n+1} - x_n}_{\text{difference}} = ax_n - x_n = (a-1)x_n = \lambda x_n. \tag{1.3}$$

The change in state—that is, the difference between the next state x_{n+1} and the current state x_n—depends on the constant $\lambda = a - 1$ and the current state x_n.

Now we turn to the continuous version of the exponential growth/decay system. We will denote the state of the system at time t by $x(t)$ and the initial state, as above, by $x_0 = x(0)$. The change in the state of the system still depends on the current state and a constant, but as the change is considered to be continuous, we use a derivative rather than a difference. The growth/decay equation (1.3) then becomes

$$x'(t) := \frac{dx}{dt} = \lambda\, x(t)\,. \tag{1.4}$$

To convert the time evolution rule into a more "useful" form, we solve the differential equation. Equation (1.4) becomes

$$\frac{x'(t)}{x(t)} = \lambda. \tag{1.5}$$

Now we can integrate both sides of (1.5) with respect to t. Notice we will get two constants of integration, one on the left and one on the right.

$$\begin{aligned} \int \frac{x'(t)}{x(t)}\, dt &= \int \lambda\, dt \\ \log|x(t)| + C_0 &= \lambda t + C_1 \\ \log|x(t)| &= \lambda t + C \\ x(t) &= e^{\lambda t + C} \\ x(t) &= De^{\lambda t}, \end{aligned}$$

where $D = e^C$ is some constant. Setting $t = 0$ we see that $x_0 = x(0) = D$, so we have

$$x(t) = x_0 e^{\lambda t}.$$

This form allows us to examine the behavior of the system for different values of λ and initial states x_0. We will analyze this system like we did with the discrete system.

Case 1: $\lambda = 0$. Then $x(t) = x_0$ for all t, so the system does not change.

Case 2: $\lambda > 0$. As with the discrete case, $x = 0$ is an unstable fixed point.

- If $x_0 = 0$, then for all t, $x(t) = 0$.
- If $x_0 > 0$, then $x(t) \to \infty$ as $t \to \infty$.
- If $x_0 < 0$, then $x(t) \to -\infty$ as $t \to \infty$.

Case 3: $\lambda < 0$. This implies that $e^{\lambda t}$ is monotone decreasing, so $x(t) \to 0$ as $t \to \infty$. Thus, $x = 0$ is a stable fixed point.

The discrete and continuous versions of this system exhibit similar qualitative behavior. We summarize with Table 1.1.

Discrete	Continuous
Model: $x_n = a^n x_0$	Model: $x(t) = x_0 e^{\lambda t}$
$a = 1$: static system	$\lambda = 0$: static system
$a > 1$: exponential growth; unstable fixed point at $x = 0$	$\lambda > 0$: exponential growth; unstable fixed point at $x = 0$
$0 < a < 1$: exponential decay; stable fixed point at $x = 0$	$\lambda < 0$: exponential decay; stable fixed point at $x = 0$

Table 1.1: A comparison of the discrete and continuous models of the same exponential system.

1.1.3 The logistic map

We conclude this chapter with a very powerful and easy to describe model, called the *logistic map*. We now consider an example where the discrete and continuous versions differ quite dramatically.

Continuous version

This time, we begin with the continuous case. Set

$$x'(t) = \lambda x(t) \left(1 - x(t)\right), \tag{1.6}$$

where $x(0) = x_0 > 0$ and $\lambda > 0$.

Because Equation (1.6) is a differential equation, it does not explicitly give us a model for the system—it only describes how the models *changes* with time. To analyze our system, we again solve the differential equation; solving for the functions $x(t)$ that satisfy Equation (1.6).

We first rewrite (1.6) to get all the variables depending on t to one side:

$$\frac{x'}{x(1-x)} = \lambda. \tag{1.7}$$

Like before in Example 1.1.2, we integrated Equation (1.4) to solve the differential equation. The challenge here is slightly different: we need an antiderivative

$$\int \frac{dx}{x(1-x)}.$$

One way to solve this is via partial fraction decomposition. We will solve this integral this way; since this is covered in a calculus class, we will omit some details.

We solve for A and B such that

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}.$$

This is equivalent to solving for A and B such that

$$1 = A(1-x) + Bx,$$

and by substituting $x = 0$ we get $A = 1$; substituting $x = 1$ gives $B = 1$. Hence we have

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$$

Therefore, Equation (1.7), can be written as

$$\frac{x'}{x} + \frac{x'}{1-x} = \lambda. \tag{1.8}$$

Each of the fractions in (1.8) can now be easily integrated, so we integrate both sides with respect to t. (Since we are integrating both sides, we will just write one constant of integration on the right side.)

$$\begin{aligned}
\int \frac{x'}{x}\,dt + \int \frac{x'}{1-x}\,dt &= \int \lambda\,dt \\
\log|x| - \log|1-x| &= \lambda t + C \\
\log\left|\frac{x}{1-x}\right| &= \lambda t + C \\
\frac{x}{1-x} &= De^{\lambda t} \\
x &= (1-x)De^{\lambda t} \\
(1+De^{\lambda t})x &= De^{\lambda t} \\
x(t) &= \frac{De^{\lambda t}}{1+De^{\lambda t}}.
\end{aligned}$$

The initial condition $x(0) = x_0$ implies that

$$\begin{aligned}
x_0 = x(0) &= \frac{D}{1+D} \\
(1+D)x_0 &= D \\
x_0 &= D(1-x_0) \\
\frac{x_0}{1-x_0} &= D.
\end{aligned}$$

Putting our work together we obtain the family of solutions

$$x(t) = \frac{De^{\lambda t}}{1+De^{\lambda t}} = \frac{\left(\frac{x_0}{1-x_0}\right)e^{\lambda t}}{1+\left(\frac{x_0}{1-x_0}\right)e^{\lambda t}}, \tag{1.9}$$

which is equivalent to

$$x(t) = \frac{x_0 e^{\lambda t}}{1-x_0+x_0 e^{\lambda t}}. \tag{1.10}$$

For large positive t, Equation (1.9) implies that

$$x(t) = \frac{De^{\lambda t}}{1+De^{\lambda t}} = 1 - \frac{1}{1+De^{\lambda t}} \approx 1 - D^{-1}e^{-\lambda t}.$$

Now we look at how the system behaves depending on its initial state. Consider four different cases for positive x_0.

Case 1: $x_0 = 0$. From (1.9), $x(t) = 0$ for all t, so $x = 0$ is a fixed point.

Case 2: $0 < x_0 < 1$. By (1.9), $D > 0$, and therefore $x(t) \to 1$ as $t \to \infty$.

Case 3: $x_0 = 1$. From (1.9), $x(t) = 1$ for all t. So $x = 1$ is a fixed point.

Case 4: $x_0 > 1$. We have $D < 0$, and $x(t) \to 1$ as $t \to \infty$.

Thus, $x = 0$ is an unstable fixed point, and $x = 1$ is a stable fixed point. This can be seen in the plot in Figure 1.2.

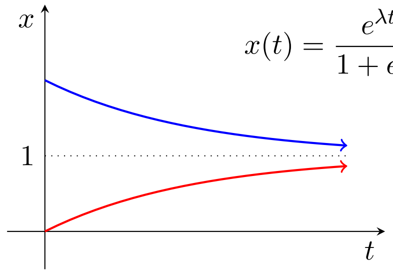


Figure 1.2: Two solutions to the logistic system with different initial conditions. When $0 < x_0 < 1$, the solution's graph looks similar to the red graph. When $x_0 > 1$, the solution's graph looks similar to the blue graph.

Discrete version

Turning now to the discrete version of the logistic map, we write a discrete version of (1.6):

$$x_{n+1} - x_n = \lambda x_n(1 - x_n). \tag{1.11}$$

We rewrite the logistic equation to get

$$\begin{aligned} x_{n+1} - x_n &= \lambda x_n(1 - x_n) \\ x_{n+1} &= x_n(\lambda + 1 - \lambda x_n) \\ x_{n+1} &= (\lambda + 1)x_n \left(1 - \left(\tfrac{\lambda}{\lambda+1}\right) x_n\right). \end{aligned}$$

Setting $y_n = \left(\frac{\lambda}{\lambda+1}\right) x_n$,

$$\begin{aligned} \left(\tfrac{\lambda+1}{\lambda}\right) y_{n+1} &= \left(\tfrac{(\lambda+1)^2}{\lambda}\right) y_n \left(1 - y_n\right) \\ y_{n+1} &= (\lambda + 1)y_n \left(1 - y_n\right), \end{aligned}$$

or when $a = \lambda + 1$,

$$y_{n+1} = (\lambda + 1)y_n(1 - y_n) = ay_n(1 - y_n). \tag{1.12}$$

The discrete version of the logistic system is nonlinear as the right side of (1.12) is quadratic in the variable y_n. This makes finding solution functions much harder than our exponential model in Section 1.1.2, but it might be a more accurate model since many phenomena are more accurately modeled with nonlinear systems.

We still want to gain *some* understanding of this system, so we look at the special case, when $a = 4$[1] and $y_n = \sin^2(\psi_n)$ for each n. Then

$$\begin{aligned} \sin^2(\psi_{n+1}) &= y_{n+1} \\ &= 4y_n(1 - y_n) \\ &= 4\sin^2(\psi_n)(1 - \sin^2(\psi_n)) \\ &= 4\sin^2(\psi_n)\cos^2(\psi_n) \\ &= \sin^2(2\psi_n). \end{aligned}$$

The equation $\sin^2(\psi_{n+1}) = \sin^2(2\psi_n)$ holds only when $\psi_{n+1} = \pm 2\psi_n + k\pi$, for some $k \in \mathbb{Z}$. We choose $\psi_{n+1} = 2\psi_n$, so $\psi_n = 2^n\psi_0$. Therefore,

$$y_n = \sin^2(2^n\psi_0). \tag{1.13}$$

Now we consider some possible initial states; note that we will not cover all possibilities.

[1]For those interested, the behavior depending on a is very chaotic.

Case 1: $\psi_0 = 0$. This gives $y_n = 0$ for all n, so $y_0 = 0$ is a fixed point.

Case 2: $\psi_0 = \frac{\pi}{2}$. This implies that $y_0 = 1$ and $y_1 = 0$. By (1.12), $y_n = 0$ for all $n \geq 1$.

Case 2′: $\psi_0 = \frac{m\pi}{2^k}$, where $m, k \in \mathbb{Z}$ such that m is odd and $k > 0$. Substituting into (1.13),

$$y_n = \sin^2\left(\frac{2^n m\pi}{2^k}\right).$$

Like in Case 2, for all $n \geq k$, $y_n = 0$, so after a finite number of steps, the system stabilizes at $y = 0$.

Case 3: $\psi_0 = \frac{m\pi}{k}$, where $m, k \in \mathbb{Z}$ such that k is odd. Notice that $\sin^2(\theta) = \sin^2(\theta + \pi)$ for all $\theta \in \mathbb{R}$. We use this fact to show that we have a *periodic* solution. If $m > k$, then let q be the largest integer such that $qk \leq m$, and let $r = m - qk$ where $0 \leq r < k$. Then

$$\sin^2\left(\frac{2^n m\pi}{k}\right) = \sin^2\left(\frac{2^n r\pi}{k} + 2^n q\pi\right) = \sin^2\left(\frac{2^n r\pi}{k}\right). \tag{1.14}$$

Therefore, this is system is equivalent to the system with initial condition $\frac{r\pi}{k}$, and $0 \leq r < k$. So without loss of generality, we assume that we start with $\frac{m\pi}{k}$, where $m < k$.

At each time step, we multiply the fraction $\frac{m}{k}$ by 2, and because of (1.14), we are only concerned with the *remainder* of $2^n m$ after dividing by k as much as possible. In other words, we are only concerned with the integer $2^n m$ modulo k[2]. Since there are only finitely many positive integers less than k, there can only be finitely many different values for y_n. Because this comes from the remainder by k, this will form a *periodic solution* . Thus, there exists a positive integer N such that for all $n \geq 0$, $y_{n+N} = y_n$. We look at two concrete cases.

Example 1.6. Let $\psi_0 = \frac{\pi}{3}$, so $m = 1$ and $k = 3$.

n	$2^n m \pmod{k}$	y_n
0	1	3/4
1	2	3/4
2	1	3/4

Even though it took us two steps to get back to the same remainder we started with, the function y_n remained constant (any ideas why?). □

Example 1.7. Let $\psi_0 = \frac{2\pi}{11}$, so $m = 2$ and $k = 11$.

n	$2^n m \pmod{k}$	y_n
0	2	$\sin^2(2\pi/11) \approx 0.29$
1	4	$\sin^2(4\pi/11) \approx 0.83$
2	8	$\sin^2(8\pi/11) \approx 0.57$
3	5	$\sin^2(16\pi/11) \approx 0.98$
4	10	$\sin^2(32\pi/11) \approx 0.08$
5	9	$\sin^2(64\pi/11) \approx 0.29$

If you correctly determined why Example 1.6 was constant, then the same reasoning can be applied here to show that the period is $N = 5$. That is, once we have made 5 steps, the values y_n will continue to repeat itself. □

[2]This is known as modular arithmetic, and remainders are written $2^n m \pmod{k}$ in this example.

Case 4: $\psi_0 = \frac{m\pi}{2^\ell k}$, where $m, k, \ell \in \mathbb{Z}$ such that $m, k, \ell > 0$ and k is odd. After ℓ steps, a "periodic orbit," also known as a cycle, is reached. (Try this for yourself.)

Case 5: $\psi_0 = c \cdot \pi$, where $c \in \mathbb{R} \setminus \mathbb{Q}$: In this case the orbits are more complicated—they may be dense.

In the discrete case the behavior varies greatly with the initial conditions and there are many more possible paths for the system. *Why is there such a distinct difference?* We will not fully answer this question, but we can already understand one of the main reasons.

We considered the continuous system

$$x'(t) = \lambda x(t)(1 - x(t))$$

and the related discrete system $y_{n+1} = ay_n(1 - y_n)$, which is equivalent to

$$x_{n+1} - x_n = \lambda x_n(1 - x_n),$$

for $a = \lambda + 1$ and $y_n = \frac{\lambda}{\lambda+1} x_n$. We looked at the case $a = 4$, that is, $\lambda = 3$. However, for the discrete system to be a good approximation of the continuous system—that is, for the difference to be a good approximation of the derivative—we need the difference, $x_{n+1} - x_n$, to be small. This is true if and only if λ is small. In our discrete example, $\lambda = 3$, which is not small enough to approximate a derivative.

Discrete version (again!)

We are coming back to the discrete version again, but now we will consider the system $y_{n+1} = ay_n(1 - y_n)$ for some fairly small values of λ. We hope to see a more pronounced connection to the continuous version, summarized in the following table.

Continuous	Discrete
λ	$a = \lambda + 1$
$x(t)$	$y_n = \frac{\lambda}{1+\lambda} x_n$
Unstable fixed point at $x = 0$	Fixed point $y = 0$ for all λ
Stable fixed point at $x = 1$	Fixed point $y = \frac{\lambda}{1+\lambda}$ for $\lambda \in (0, 1)$

Let $\lambda \in (0, 1)$, so that $a \in (1, 2)$ and $0 \leq \frac{\lambda}{\lambda+1} = \frac{a-1}{a} < \frac{1}{2}$. Define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(y) := ay(1 - y).$$

Then for $y \in (0, 1)$,

$$0 < f_a(y) \leq \frac{a}{4} < 1,$$

as the function has maximum $f_a(\frac{1}{2}) = \frac{a}{4}$. This can be seen in Figure 1.3. Observe also that

- the function f_a is strictly increasing on the interval $(0, 1/2)$ and
- the function f_a is strictly decreasing on the interval $(1/2, 1)$.

We use f_a to derive a discrete system: $y_{n+1} = f_a(y_n)$ for all $n \geq 0$.

Recall that a function $f(x)$ is **monotonically increasing**[3] if for all pairs x_1, x_2 in the domain of f, $f(x_1) \leq f(x_2)$ implies that $x_1 \leq x_2$. A similar definition follows for monotonically decreasing. Analogous definitions hold for sequences as well. The sequence $(x_n)_{n=0}^{\infty}$ is monotonically increasing if for all $n, m \geq 0$, $x_n \leq x_m$ implies that $n \leq m$.

We will briefly analyze the discrete system for different initial conditions $y_o \in [0, (a - 1)/a]$.

[3] In case you want more details.

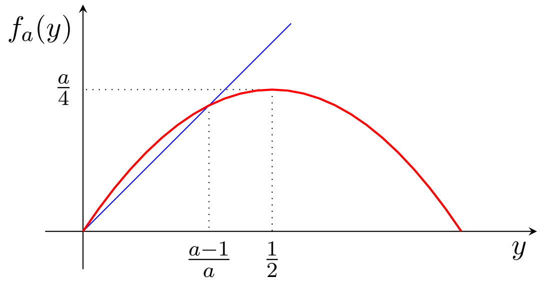


Figure 1.3: The plot of the function $f_a(y)$ for some $a \in (1,2)$ is given in red, and the plot of $g(y) = y$ is given in blue.

Case 1: $y_0 = 0$. This implies that $y_n = 0$ for all n, so $y = 0$ is a fixed point.

Case 2: $y_0 = \frac{a-1}{a}$. This condition implies that

$$y_1 = f_a(y_0) = a\left(\frac{a-1}{a}\right)\left(1 - \frac{a-1}{a}\right) = \frac{a-1}{a}.$$

Thus, $y = \frac{a-1}{a}$ is also a fixed point.

Case 3: $y_0 \in (0, (a-1)/a)$. We will prove that y_n is monotonically increasing and $y_n \to (a-1)/a$ as $n \to \infty$.

Case 4: $\frac{a-1}{a} < y_0 < \frac{1}{2}$. This will be proved in the homework and is similar to Case 3.

To prove the claim in Case 3, we split up the proof into a few lemmas.

Lemma 1.8. *In Case 3, for all $n \geq 0$,*

$$0 < y_n < \frac{a-1}{a}. \tag{1.15}$$

Proof. Since $0 < f_a(y) < 1$ for $0 < y < 1$, it follows that $0 < y_n$. So we just need to prove the second inequality: $y_n < (a-1)/a$, for all n. We do this by induction on n. For the base case, $n = 0$,

$$y_0 < \frac{a-1}{a}$$

which follows from our assumption in Case 3. Now assume that (1.15) holds for n, and we will show that it holds for $n+1$. Recall that f_a is monotonically increasing on $(0, (a-1)/a)$. Therefore,

$$\begin{aligned} y_{n+1} &= f_a(y_n) && \text{(definition)} \\ &< f_a\left(\frac{a-1}{a}\right) && \left(\begin{array}{l}\text{induction step} \\ \text{and monotone}\end{array}\right) \\ &= \frac{a-1}{a}. && \text{(evaluate)} \end{aligned}$$

Thus, $y_{n+1} < (a-1)/a$, and so by induction this holds for all $n \geq 0$. □

Lemma 1.9. *In Case 3, the sequence $(y_n)_{n=0}^{\infty}$ is monotonically increasing.*

Proof. By Lemma 1.8, $-y_n > -(a-1)/a$, so

$$\begin{aligned} y_{n+1} &= a y_n (1 - y_n) \\ &> a y_n \left(1 - \frac{a-1}{a}\right) \\ &= y_n. \end{aligned}$$

□

Proposition 1.10. *In Case 3,*

$$\lim_{n\to\infty} y_n = \frac{a-1}{a}.$$

Proof. From Lemmas 1.8 and 1.9, the sequence (y_n) is monotonically increasing and bounded above. Therefore, the limit exists.

Let

$$y_* := \lim_{n\to\infty} y_n.$$

Since f_a is continuous,

$$y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} f_a(y_n) = f\left(\lim_{n\to\infty} y_n\right) = f_a(y_*).$$

Hence y_* is a fixed point, and thus

$$y_* = ay_*(1-y_*).$$

Therefore, we have two different cases: either $y_* = 0$ or $1 = a(1-y_*)$. But by Lemma 1.8, $y_n > 0$ for all n, and by Lemma 1.9, (y_n) is increasing. Thus, it is not possible that $y_* = 0$. Hence, $1 = a(1-y_*)$, so

$$y_* = \frac{a-1}{a}.$$

□

To finish out the final case, Case 4, we prove similar statements that we proved above.

Proposition 1.11. *In Case 4, for all $n \geq 0$,*

(i) $\dfrac{a-1}{a} < y_n < \dfrac{1}{2}$,

(ii) the sequence (y_n) is monotone decreasing, and

(iii) $\lim_{n\to\infty} y_n = \dfrac{a-1}{a}$.

Proof. Exercise. □

From Propositions 1.10 and 1.11, for an initial state y_0 in the interval $(0, 1/2)$, we have $y_n \to \frac{a-1}{a}$. Thus for $1 < a < 2$, $y = \frac{a-1}{a}$ is a stable fixed point of the system and $y = 0$ is an unstable fixed point, which matches more closely the continuous version of the logistic map.

Chapter 2

Discrete dynamical systems

We begin with some basic notation and definitions. As usual, the composition of functions $f, g : \mathbb{R}^d \to \mathbb{R}^d$ is denoted by $f \circ g$, where $f \circ g(x) := f(g(x))$. The identity function is denoted by id, and it always satisfies $\mathrm{id}(x) = x$ for all x. The composition of f with itself n times is

$$f^n := \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}},$$

for $n \in \mathbb{N}$, and $f^0 := \mathrm{id}$. If the inverse of f exists, we denote it by f^{-1}. That is, f^{-1} is the function such that $f^{-1} \circ f = f \circ f^{-1} = \mathrm{id}$. Then we have

$$f^{-n} = \underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{n \text{ times}}.$$

Note that $f^n(x) = f(f(\ldots f(x) \ldots)) \neq (f(x))^n$.

Definition 2.1. Let $X \subseteq \mathbb{R}^d$ and $f : X \to X$. The expression

$$x_{n+1} = f(x_n) \tag{2.1}$$

is called a **difference equation** . The set X is the **state space** (or phase space) of the system, and any sequence $(x_n)_{n=0}^{\infty}$ satisfying (2.1) is called a **solution** of (2.1). If the expression $f(x_n)$ is independent of n, that is, if the rule (2.1) is independent of the time, then the system is called a discrete **autonomous dynamical system** .

Note that the function f may depend on extra parameters from some set Y. That is, in general $f : Y \times X \to X$. We saw this in Section 1.1.3 for example.

Example 2.2.

(i) The discrete exponential growth/decay system of Section 1.1.2 is an autonomous system as we have $x_{n+1} = ax_n$. In other words, $f(x) := ax$, and the value of f depends on a value from the state space, x, and on another parameter, a, but is independent of n.

(ii) An example of a non-autonomous system is

$$x_{n+1} := (n+1)x_n.$$

Here, rather than one function f, we have a system of functions: $f_n(x) = (n+1)x$, and

$$x_n = nx_{n-1} = n(n-1)x_{n-2} = \ldots = n!x_0.$$

Remark 2.3. For the next two points, we assume that (2.1) defines an autonomous system.

(i) For any initial state x_0, we have the sequence $x_0, x_1 = f(x_0), x_2 = f(x_1)$, and so on. Thus for any initial state x_0, a solution is given by

$$(f^0(x_0), f(x_0), f^1(x_0), f^2(x_0), \ldots).$$

The sequence $(\mathrm{id}, f, f^2, \ldots)$ itself may sometimes be referred to as the dynamical system.

(ii) For $m, n \in \mathbb{Z}$, we have

$$f^{n+m}(x) = f^n(f^m(x)) = f^m(f^n(x)).$$

Hence if $x_0 \in X$ and we set $x_n := f^n(x_0)$ for all $n \in \mathbb{N}$, then for all $n, m \in \mathbb{N}_0$ we have

$$f^m(x_n) = x_{m+n} = f^n(x_m).$$

This is sometimes referred to as the *semigroup property*[1].

2.1 Discrete autonomous dynamical systems

Throughout this section, we let $X \subseteq \mathbb{R}^d$, $f : X \to X$, and the discrete dynamical system defined by $x_{n+1} = f(x_n)$ is autonomous.

Definition 2.4. Let $x \in X$.

(i) The sequence

$$O^+(x) := \big(x, f(x), f^2(x), \ldots\big)$$

is called the **forward orbit** (or just the **orbit**) of x.

(ii) Assuming f is bijective, the sequence

$$O^-(x) := \big(x, f^{-1}(x), f^{-2}(x), \ldots\big)$$

is called the **backward orbit** of x.

(iii) Assuming f is bijective, the **full orbit** of x is the (two-way) sequence

$$O(x) := (f^n(x))_{n=-\infty}^{\infty}.$$

Example 2.5.

(i) Consider again the system given by the function $f(x) = ax$, for all $x \in \mathbb{R}$. For $a \neq 0$, f is bijective, with inverse $f^{-1}(x) = \frac{x}{a}$. The three orbits are

$$\begin{aligned} O^+(x) &= \big(x, ax, a^2x, \ldots\big) = \big(a^n x\big)_{n=0}^{\infty}, \\ O^-(x) &= \left(x, \frac{x}{a}, \frac{x}{a^2}, \ldots\right), \\ O(x) &= \big(a^n x\big)_{n=-\infty}^{\infty}. \end{aligned}$$

(ii) Not all functions have a backward orbit on X or even at all. For example, the function $f(x) = ax(1-x)$. In this case, f is not invertible, as $f(0) = f(1) = 0$.

Definition 2.6. A point $x \in X$ is called a **fixed point** of the dynamical system $x_{n+1} = f(x_n)$ if $x = f(x)$. A point $x \in X$ is called a **periodic point** of period m if $f^m(x) = x$. The smallest integer $p > 0$ such that $f^p(x) = x$ is called the **minimum period** (also called minimal period, least period, and prime period) of x.

[1] This is because the solutions form a semigroup.

Note that a fixed point is exactly a periodic point of period 1. And more generally, a point x is periodic of period m if and only if x is a fixed point of the function f^m.

Lemma 2.7. *If $x \in X$ is a periodic point of period m, then x is a periodic point of period mn, for all $n \in \mathbb{N}$. Conversely, if x is periodic with minimal period p and $f^m(x) = x$ for some $m > 0$, then $p \mid m$.*

Proof. Let $x \in X$ be periodic of period m. We prove the first statement by induction. The base case, $n = 1$, simply says that $f^m(x) = x$, which follows from the definition. Now suppose that for some $n \in \mathbb{N}$, $f^{mn}(x) = x$. Then x is a fixed point of $f^{m(n+1)}$ as

$$f^{m(n+1)}(x) = f^{mn+m}(x) = f^{mn}(f^m(x)) = f^{mn}(x) = x.$$

Conversely, we suppose that x has minimum period p. By the division algorithm, there exist integers $q, r \geq 0$ with $r < p$ such that $m = pq + r$. Thus,

$$x = f^m(x) = f^{pq+r}(x) = f^r(f^{pq}(x)) = f^r(x).$$

Since $r < p$ and p is minimal, we must have $r = 0$. That is, $m = pq$, and so $p \mid m$. $\square$

If a point x is periodic, then all points appearing in the orbit must also be periodic, with the same period (but not necessarily the same minimum period). This is what the next proposition proves.

Proposition 2.8. *If $x \in X$ is a periodic point of period m, then for all $q \in \mathbb{N}$, $f^q(x)$ is a periodic point of period m.*

Proof. Let $m \in \mathbb{N}_0$, $x \in X$ such that $f^m(x) = x$. Then for $q \in \mathbb{N}$,

$$f^m(f^q(x)) = f^{m+q}(x) = f^{q+m}(x) = f^q(f^m(x)) = f^q(x). \qquad \square$$

Since all points in the orbit of x are periodic when x is periodic by Proposition 2.8, this makes the notion of **periodic orbits** well-defined.

Definition 2.9. A point $x \in X$ is called **eventually periodic** of period p if x is not periodic and there exists an $m > 0$ such that $f^m(x)$ is periodic of period p.

Sometimes it is useful to define eventually periodicity to include (actual) periodicity—that is, to allow $m = 0$ in the definition. We choose however to distinguish the two concepts, so to avoid confusion. For an eventually periodic point x_0, the forward orbit looks like

$$O^+(x_0) = (x_0, x_1, \ldots, x_{m-1}, \underbrace{x_m, x_{m+1}, \ldots, x_{m+p-1}}, \underbrace{x_m, x_{m+1}, \ldots x_{m+p-1}}, \ldots).$$

We may also refer to an eventually periodic point as having an **eventually periodic orbit** since periodic orbits are well-defined.

Example 2.10. Consider the function $f(x) = ax(1 - x)$. The point $x = 1$ has forward orbit

$$O^+(1) = \big(1, 0, 0, \ldots\big),$$

which is eventually periodic of period 1. In other words, it is eventually fixed. $\square$

Theorem 2.11. *Let $f : X \to X$ where X is a finite set. Then for any $x \in X$, the orbit under f is either periodic or eventually periodic.*

Proof. Let $|X| = n$ and consider the first $n + 1$ elements in the orbit of some $x \in X$,

$$x, f(x), f^2(x), \ldots, f^n(x).$$

By the pigeonhole principle, we must have $f^i(x) = f^j(x)$ for some integers $0 \leq i < j \leq n$. Thus $f^i(x)$ is a periodic point of period $p = j - i$, and by Proposition 2.8, so is $f^k(x)$ for all $k \geq i$. $\square$

Similarly, if the state space is countable, then there are not many options for orbits. (Recall the cardinality of X is countable if there exists a bijection of sets $\mathbb{N} \to X$.)

Theorem 2.12. *Let $f : \mathbb{N} \to \mathbb{N}$. Then for $x \in \mathbb{N}$, exactly one of the following is true.*

1. *The orbit of x is periodic or eventually periodic.*
2. *The orbit of x diverges*[2].

Proof. Assume that $O^+(x)$ does not diverge, so we want to show that the orbit is either periodic or eventually periodic. Then there is an $r \in \mathbb{N}$ such that for all $n \geq 0$, $f^n(x) < r$. Therefore, there are more than r values $f^n(x)$ within the interval $[0, r]$. Then, by the pigeonhole principle, there are distinct $m, n \in \mathbb{N}$ such that $f^m(x) = f^n(x) < r$. Assuming $m < n$, all elements $f^n(x)$ for $n \geq m$ must be periodic, so the orbit $O^+(x)$ is either periodic or eventually periodic. □

Theorem 2.13. *Let $f : \mathbb{Z} \to \mathbb{Z}$. Then for $x \in \mathbb{Z}$, exactly one of the following is true.*

1. *The orbit of x is periodic or eventually periodic.*
2. *The orbit of x diverges.*

Proof. Exercise. (The argument is essentially the same as above, but using $|f^n(x)|$ instead of $f^n(x)$. □

Remark 2.14. It is not always easy to decide which of the two cases in Theorems 2.12 or 2.13 an orbit falls into. A classical example of this is the so-called "$3n + 1$ problem," in which a dynamical system is specified by the function $f : \mathbb{N} \to \mathbb{N}$ where

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even.} \end{cases}$$

For fairly small initial values (like integers $\leq 10^{18}$), the orbits can be found by computer calculations and are periodic or eventually periodic. But to this day, it is not known whether or not this is true for all initial values. It is a famous open conjecture, known as the Collatz conjecture, that asks if all positive integers converge to 1.

One of the challenging aspects of the Collatz conjecture and deciding whether or not an orbit will converge are related to a famous problem in mathematics and computer science: the Halting Problem. The problem asks, given an arbitrary computer program and input, decide if it will halt and return an answer. Turing, in 1936, proved that there cannot exist an algorithm that can successfully decide whether or not every possible program and input pair halts or not.

For $x \in \mathbb{R}$, we write $\|x\| = |x|$ the standard absolute value[3].

Definition 2.15. A function $f : X \to X$ is said to be a **contraction** if there exists a constant $K \in [0, 1)$ such that for all $x, y \in X$,

$$\|f(x) - f(y)\| \leq K \|x - y\| .$$

We prove a quick lemma to show that all functions that are contraction are continuous functions.

Lemma 2.16. *If a function $f : X \to X$ is a contraction, then f continuous.*

[2] A sequence diverges if it does not converge to any real number.
[3] This is one of many norms we could use.

Proof. If the contraction constant $K = 0$, then this is clear as f is constant. Otherwise, let $x_0 \in X$. For all $\varepsilon > 0$, set $\delta = \varepsilon/K$. Then for all $x \in D$ such that $\|x - x_0\| < \delta$, we have

$$\|f(x) - f(x_0)\| \leq K \|x - x_0\| < K\delta = \varepsilon.$$

Therefore, f is continuous at x_0, and so f is continuous on X. $\square$

Theorem 2.17 (Banach's fixed-point theorem)**.** *If $f : X \to X$ is a contraction with $X \subseteq \mathbb{R}^d$ closed, then f has a unique fixed point $\overline{x} \in X$ and the orbit of every $x \in \mathbb{R}$ converges to $\overline{x}$.*

Proof. First we prove that an arbitrary point in X converges to some point in X. Let $x_0 \in X$, and for every $n \in \mathbb{N}$ define $x_n = f^n(x_0)$. For each $n \in \mathbb{N}$,

$$x_n = \sum_{i=0}^{n-1}(x_{i+1} - x_i) + x_0,$$

so that the sequence $(x_n)_{n=0}^{\infty}$ converges if and only if the series $\sum_{i=0}^{\infty}(x_{i+1} - x_i)$ converges. Since f is a contraction, there exists $K \in [0, 1)$ such that

$$\|x_{i+1} - x_i\| = \|f(x_i) - f(x_{i-1})\| \leq K \|x_i - x_{i-1}\|. \tag{2.2}$$

By iterating (2.2), $\|x_{i+1} - x_i\| \leq K^i \|x_1 - x_0\|$. Therefore,

$$\begin{aligned}
\left\| \sum_{i=0}^{\infty}(x_{i+1} - x_i) \right\| &\leq \sum_{i=0}^{\infty} \|x_{i+1} - x_i\| && \text{(triangle inequality)} \\
&\leq \sum_{i=0}^{\infty} K^i \|x_1 - x_0\| && \text{(contraction)} \\
&= \|x_1 - x_0\| \sum_{i=0}^{\infty} K^i && \text{(constant)} \\
&< \infty. && \text{(geometric series)}
\end{aligned}$$

Because the series converges, the sequence also converges to a point $\bar{x} := \lim_{n\to\infty} x_n$. Since X is closed and since $\mathbb{R}^d$ is complete, it follows that $\bar{x} \in X$.

Now we want to prove that $\bar{x}$ is unique. Since f is a contraction, by Lemma 2.16, f is continuous. Thus,

$$\bar{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(\bar{x}).$$

This proves that $\bar{x}$ is a fixed point of f. Suppose there is another fixed point of f, call it $\bar{y}$. We will show $\bar{y} = \bar{x}$, which will prove uniqueness. Since $\bar{y}$ is a fixed point,

$$\|\bar{x} - \bar{y}\| = \|f(\bar{x}) - f(\bar{y})\| \leq K \|\bar{x} - \bar{y}\|. \tag{2.3}$$

However, since $K \in [0, 1)$, (2.3) is only possible when $\|\bar{x} - \bar{y}\| = 0$. This implies that $\bar{y} = \bar{x}$. $\square$

From Banach's fixed-point theorem, we see not only that a contraction map must have a fixed point on a closed set X, but that the orbit of every point in the space converges to the fixed point. In the proof, we have also shown the following converse statement, which is true for all continuous functions, not just contractions.

Corollary 2.18. *Let $f : X \to X$ be continuous. If the orbit of $x \in X$ converges to some $y \in X$, then y is a fixed point of f.*

2.2 Fixed points

We now consider some properties of fixed points in more detail, some of which we alluded to in the previous section.

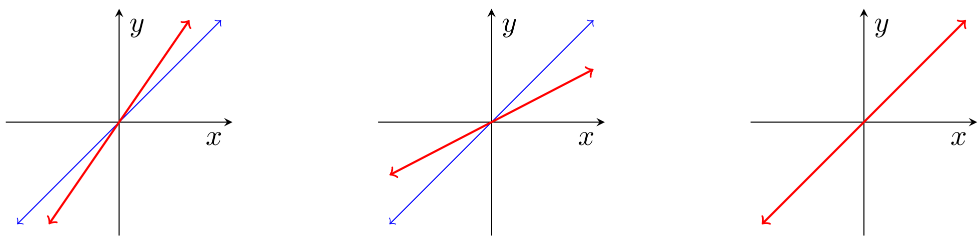
Definition 2.19. Let $f : X \to X$.

1. The point x is a **stable fixed point** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for $y \in X$, $\|x - y\| < \delta$ implies that for all $n \geq 0$,
$$\|f^n(y) - x\| < \varepsilon.$$
2. The point x is an **attracting fixed point** (or **sink**) if there exists $\delta > 0$ such that for $y \in X$, $\|x - y\| < \delta$ implies that
$$\lim_{n\to\infty} f^n(y) = x.$$
3. The point x is an **unstable fixed point** if x is not stable.
4. The point x is a **repelling fixed point** (or **source**) if there exists $\delta > 0$ such that for $y \in X$ with $0 < \|x - y\| < \delta$ there exits an $m \geq 0$ such that
$$\|x - f^m(y)\| > \delta.$$

Qualitatively, if x is a stable fixed point, we can make a system stay arbitrarily close to x "forever" by choosing the initial state close enough to x. If x is an attracting fixed point, then we can make a system converge to the state x by choosing an initial state close enough to x. If f is continuous and $X \subseteq \mathbb{R}$, then all attracting fixed points are stable (see Proposition 2.24). A fixed point is repelling if we can find some ball around x such that for all points in the ball, their orbit at some point leaves the ball. Thus all repelling fixed points are unstable.

We first go through a number of examples to demonstrate the subtleties.

Example 2.20. Consider again the discrete system of Example 1.3, defined by $x_{n+1} = a \cdot x_n = f(x_n)$ for some $a > 0$. Therefore, $f(x) = ax$, and note that we always have at least one fixed point: $x = 0$. We now use the function f to reexamine the qualitative behavior of this system.[4]



(a) Case 1: $a > 1$. (b) Case 2: $a < 1$. (c) Case 3: $a = 1$.

Figure 2.1: The above pictures show the function f in red, which we use to examine the path of the system over time, and the identity function id in blue.

Case 1: $a > 1$. As $f(x) > x$ for $x > 0$ and $f(x) < x$ for $x < 0$, the sequence (x_n) is increasing (away from zero) if $x_0 > 0$ and decreasing (away from zero) if $x_0 < 0$. In other words, for every initial state other than zero, the system diverges away from zero, and thus $x = 0$ is a repelling fixed point.

[4] Note that a graph of the path system $x_n = a^n x_0$ as a function of n, looks very different from this.

Case 2: $a < 1$. Here, the opposite is true: $f(x) < x$ for $x > 0$ and $f(x) > x$ for $x < 0$. So (x_n) is decreasing (towards zero) if $x_0 > 0$ and increasing (towards zero) if $x_0 < 0$. Thus for every initial state other than zero, the system converges back to zero, and thus $x = 0$ is an attracting fixed point.

Case 3: $a = 1$. We have $f(x) = x$ for all $x \in \mathbb{R}^d$. Therefore every point is a fixed point. Each fixed point x is stable, but neither attracting nor repelling: given an initial state, y, arbitrarily close to x, the system will stay in state y "forever," and thus not get further away from or closer to x. Formally, for $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$ we have

$$\|x - f^n(y)\| = \|x - y\|\,,$$

so for $\varepsilon > 0$, $\|x - f^n(y)\| < \varepsilon$ whenever $\|x - y\| < \varepsilon =: \delta$. □

The above one-dimensional system is defined by a differentiable function and the analysis gives us a hint as to how to classify fixed points using derivatives. We will look more closely at this in the next chapter.

Example 2.21. Consider the function g in Figure 2.2.

$$g(x) = \begin{cases} 2x & \text{if } x \geq 0, \\ \frac{1}{2}x & \text{if } x < 0. \end{cases}$$

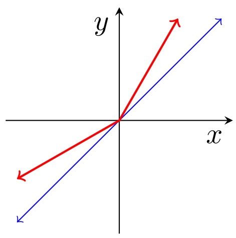


Figure 2.2: The function g, plotted in red, and the identity function, plotted in blue.

Then g has one fixed point at $x = 0$, and for all $x \neq 0$, $g(x) > x$, which means that the orbit of x is increasing. However, for $x_0 < 0$, we have $0 > x_{n+1} > x_n$, which is increasing towards zero. For $x_0 > 0$, we have $0 < x_n < x_{n+1}$, which is increasing away from zero. Thus, the fixed point $x = 0$ is not stable, attracting, or repelling, so it is just unstable. □

Example 2.22. Consider the two functions f and g in Figure 2.3

$$f(x) = \begin{cases} -3x & \text{if } x < 0, \\ -\frac{1}{2}x & \text{if } x \geq 0. \end{cases} \qquad g(x) = \begin{cases} -2x & \text{if } x < 0 \\ -\frac{1}{3}x & \text{if } x \geq 0\,. \end{cases}$$

Figure 2.3: The functions f and g are plotted in red together with the identity and its negative are plotted in blue.

It is clear that both f and g have the fixed point at $x = 0$, and $f(x) > -x$, $g(x) > -x$ for all $x \neq 0$. This has not told us much to distinguish f and g, so instead we look at a few iterations. Fix $x_0 < 0$, so the first few iterations are

$$x_1 = f(x_0) = -3x_0, \qquad x_2 = -\frac{1}{2}x_1 = \frac{3}{2}x_0, \qquad x_3 = -\frac{9}{2}x_0, \qquad x_4 = \frac{9}{4}x_0.$$

Therefore,

$$x_{2n} = \left(\frac{3}{2}\right)^n x_0, \qquad x_{2n+1} = -3\left(\frac{3}{2}\right)^n x_0.$$

Similarly, if $x_0 > 0$,

$$x_{2n} = \left(\frac{3}{2}\right)^n x_0, \qquad x_{2n+1} = -\frac{1}{2}\left(\frac{3}{2}\right)^n x_0.$$

Thus, the point $x = 0$ is a repelling fixed point of f.

In the same way, we find that for a point $x_0 < 0$ under g

$$x_{2n} = \left(\frac{2}{3}\right)^n x_0, \qquad x_{2n+1} = -2\left(\frac{2}{3}\right)^n x_0.$$

For $x > 0$,

$$x_{2n} = \left(\frac{2}{3}\right)^n x_0, \qquad x_{2n+1} = -\frac{1}{3}\left(\frac{2}{3}\right)^n x_0.$$

We see that the point $x = 0$ is an attracting fixed point of g.

As $g : \mathbb{R} \to \mathbb{R}$ is continuous, this implies that the fixed point is stable. To see this directly, let $\varepsilon > 0$. We need to find a $\delta > 0$ such that $\|y\| < \delta$ implies that for all $n \geq 0$, $\|g^n(y)\| < \varepsilon$. To this end, set $\delta = \varepsilon/2$, so that whenever $\|y\| < \delta$, then

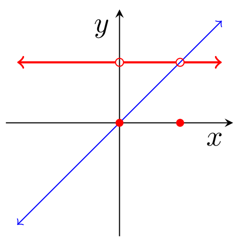
$$\|g^n(y)\| \leq \begin{cases} (2/3)^m \|y\| & \text{if } n = 2m, \\ 2 \cdot (2/3)^m \|y\| & \text{if } n = 2m+1. \end{cases}$$

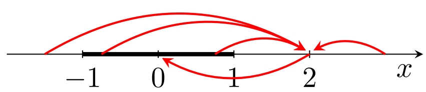
Therefore, $\|g^n(y)\| \leq 2\|y\| < 2\delta = \varepsilon$. □

We've seen that a fixed point can be neither repelling nor attracting. It is also possible for a fixed point to be both repelling and attracting, which is demonstrated in Example 2.23.

Example 2.23. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 2, \\ 2 & \text{otherwise.} \end{cases}$$





The function f has one fixed point at $x = 0$. To see that it is attracting, observe that for any $x \in \mathbb{R}$ we have $f^2(x) = 0$, so the orbit of every point in $\mathbb{R}$ is eventually fixed and thus converges to 0. To see that it is also repelling, take $\delta = 1$. Then for every $y \neq 0$ with $\|y - 0\| < 1$ that $\|f(y) - 0\| = 2 > 1$. This also shows that the fixed point $x = 0$ is not stable. □

If f is a continuous function (on $\mathbb{R}$), then such a situation in Example 2.23 cannot occur, which the next proposition proves.

Proposition 2.24. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with an attracting fixed point $\bar{x}$, then $\bar{x}$ is a stable fixed point of the system $x_{n+1} = f(x_n)$.*

Proof. From the definition of an attractor, there exists a maximal interval $I := (a, b)$ containing $\bar{x}$ such that for all $x \in I$,

$$\lim_{n\to\infty} f^n(x) = \bar{x}.$$

We allow for the possibility that $a = -\infty$ or $b = \infty$. By construction, I cannot contain any other fixed points, and additionally, for all $x \in I \setminus \{\bar{x}\}$, exactly one of the inequalities holds:

$$f(x) < x \qquad\qquad f(x) > x.$$

Let $I_\ell := (a, \bar{x})$ and $I_r := (\bar{x}, b)$ be the left and right sides of I, respectively. Our goal is to prove the following claim.

Claim 1. For all $x \in I_\ell$, the orbit of x is monotonically increasing, and for all $x \in I_r$, the orbit of x is monotonically decreasing.

Proof of Claim 1. Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := f(x) - x$. Since f is continuous, so is g. Note that f cannot have a periodic point contained in $I \setminus \{\bar{x}\}$; otherwise such a point would not converge to $\bar{x}$. Therefore, g has exactly one root in the interval I, namely $\bar{x}$, so g has the same sign on I_r and on I_ℓ (can be different signs).

Suppose now that $g > 0$ on I_r. That is, for all $x \in I_r$, $f(x) > x$. Now we iterate this fact: for all $x \in I_r$ and $n \geq 1$,

$$f^{n+1}(x) = f(f^n(x)) > f^n(x) > x.$$

Thus, the sequence $(f^n(x))_{n=0}^\infty$ is a monotonically increasing sequence in I_r. Since $g > 0$ on I_r, it follows that for all $x \in I_r$, $x > \bar{x}$. Therefore the sequence $(f^n(x))_{n=0}^\infty$ cannot converge to $\bar{x}$—a contradiction. Hence, we must have that $g < 0$ on I_r. A similar argument shows that $g > 0$ for all $x \in I_\ell$. Therefore, we have proved Claim 1. □

Now we are ready to show that $\bar{x}$ is a stable fixed point. Let $\varepsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that $\|x - \bar{x}\| < \delta$ implies $\|f(x) - \bar{x}\| < \varepsilon$. In particular, we may choose δ such that $\delta \leq \varepsilon$. Fix this choice of δ, and let $x \in I \setminus \{\bar{x}\}$ such that $\|x - \bar{x}\| < \delta$. Then either $x \in I_\ell$ or $x \in I_r$; the argument will work the similarly either way, so we choose that $x \in I_r$. Therefore, $\|x - \bar{x}\| = x - \bar{x} > 0$. From Claim 1, it follows that $x > f(x) > f^2(x) > \cdots > \bar{x}$. Thus, for all $n \geq 0$,

$$\begin{aligned}
\|f^n(x) - \bar{x}\| &= f^n(x) - \bar{x} && (f^n(x) \in I_r)\\
&< x - \bar{x} && (\text{Claim } 1)\\
&= \|x - \bar{x}\| && (x \in I_r)\\
&< \delta \leq \varepsilon. && (\text{definition})
\end{aligned}$$

Therefore, $\bar{x}$ is a stable fixed point. □

In the next example, we will use polar coordinates. Recall that polar coordinates are related to the usual Cartesian coordinates by $x = r\cos(t)$ and $y = r\sin(t)$. We will use $r \geq 0$ and $t \in [0, 2\pi)$ in this way for the next example. The benefit of polar coordinates is that, for some functions, polar coordinates are easier to work with.

Example 2.25. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined, in polar coordinates, by

$$f(r, t) := (\sqrt{r}, \sqrt{2\pi t}).$$

The function f is continuous and has fixed points at the origin $(0, 0)$ and $(1, 0)$. For an initial point (r_0, t_0), we have

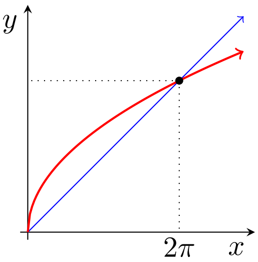
$$r_1 = r_0^{1/2}, \qquad r_2 = \left(r_0^{1/2}\right)^{1/2} = r_0^{1/4}, \qquad r_n = r_0^{1/2^n}$$

and

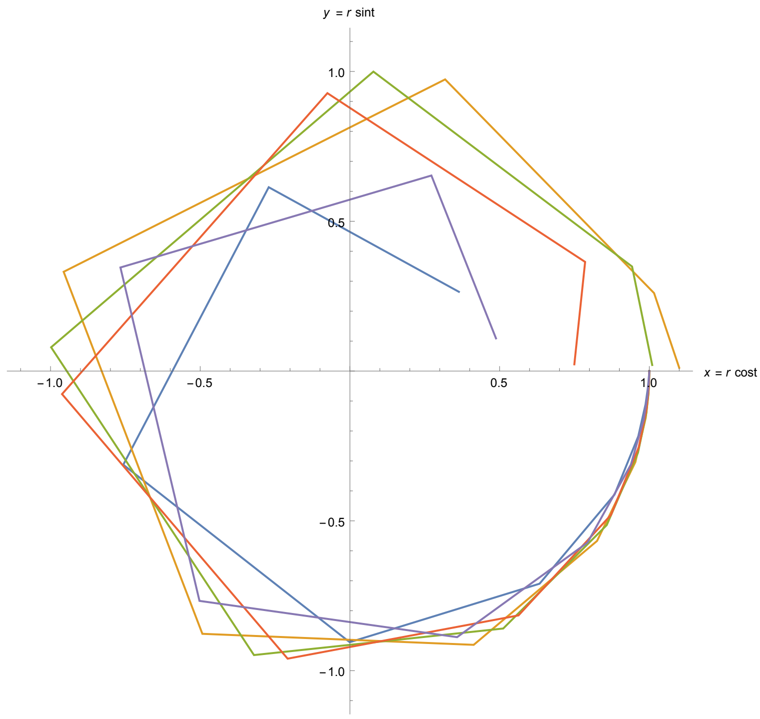
$$t_1 = (2\pi t)^{1/2}, \qquad t_2 = \left(2\pi(2\pi t)^{1/2}\right)^{1/2} = (2\pi)^{3/4} t^{1/4}, \qquad t_n = (2\pi)^{(2^n-1)/2^n} t^{1/2^n}.$$

Thus as $n \to \infty$, both $r_n \to 1$ and $t_n \to 2\pi$, and we see that the orbit of every (r, t) converges to $(1, 0)$. Hence the fixed point $(1, 0)$ is attracting.

However, $(1, 0)$ is also unstable. If we take any point in the first quadrant of the plane—so $r > 0$ and $0 < t < \pi/2$—the orbit of the point takes an anticlockwise path around the origin before approaching $(1, 0)$ from the fourth quadrant. We can see this by examining the behavior of the t-function $f_1 : \mathbb{R} \to \mathbb{R}$ via $f_1(t) = \sqrt{2\pi t}$ on the interval $(0, 2\pi)$. For all $x \in (0, 2\pi)$, f_1 is increasing, with $f_1(x) > x$ and $0 < f_1(x) < 2\pi$. □



(a) The function f_1 is plotted in red and the identity is plotted in blue.



(b) The trajectories of the the points: $(1.1, 0.01)$, $(1.01, 0.02)$, $(0.75, 0.03)$, $(0.5, 0.22)$, and $(0.45, \pi/5)$.

Figure 2.5: Plots concerning the system in Example 2.25.

2.3 Periodic orbits

Definition 2.26. Let x be a fixed point of a function $f : X \to X$. A point $y \in X$ is called **forward asymptotic** to x if

$$\lim_{k\to\infty} f^k(y) = x.$$

The set of all points forward asymptotic to x is called the **stable set** of x and is denoted by $\omega^s(x)$.

Observe that if x is an attracting fixed point then the set $\omega^s(x)$ contains a ball of radius δ around x.

Definition 2.27. A periodic orbit $O^+(x)$ of period p is a **stable orbit** if each of its points

$$x, f(x), f^2(x), \ldots, f^{p-1}(x)$$

is a stable fixed point of f^p. A periodic orbit is called **unstable** if it is not stable.

Proposition 2.28. *Let $f : X \to X$ be continuous. Then a periodic orbit $O^+(x)$ of period p is stable if and only if x is a stable fixed point of f^p.*

Proof. Let $x \in X$ such that the orbit $O^+(x)$ is a stable orbit of period p. By definition, x is a stable fixed point of f^p, which proves the forward direction.

Conversely, assume that $x \in X$ is a stable fixed point of f^p. We need to show that $f^i(x)$ is a stable fixed point of f^p for all $0 \le i \le p-1$. Then let $0 \le i \le p-1$. Since f is continuous, so is f^i. Therefore, for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $\|x - y\| < \delta_1$ implies

$$\left\| f^i(x) - f^i(y) \right\| < \varepsilon.$$

Since x is a stable fixed point of f^p, there exists $\delta_2 > 0$ such that $\|x - y\| < \delta_2$ implies that for all $k \in \mathbb{N}$,

$$\left\| x - f^{kp}(y) \right\| < \delta_2.$$

By the continuity of f^{p-i}, there exists a $\delta_3 > 0$ such that $\|u - z\| < \delta_3$ implies that

$$\left\| f^{p-i}(u) - f^{p-i}(z) \right\| < \delta_2.$$

Now set $u = f^i(x)$. Recall that $f^p(x) = x$, so for all $k \in \mathbb{N}$,

$$\begin{aligned}
\left\| f^i(x) - z \right\| < \delta_3 &\Longrightarrow \left\| f^p(x) - f^{p-i}(z) \right\| < \delta_2 && \left(\text{continuity of } f^{p-i}\right) \\
&\Longrightarrow \left\| x - f^{kp+p-i}(z) \right\| < \delta_1 && \left(\text{fixed point of } f^p\right) \\
&\Longrightarrow \left\| f^i(x) - f^{kp+p}(z) \right\| < \varepsilon && \left(\text{continuity of } f^i\right) \\
&\Longrightarrow \left\| f^i(x) - f^{(k+1)p}(z) \right\| < \varepsilon.
\end{aligned}$$

Therefore, $f^i(x)$ is a stable fixed point of f^p for all $0 \le i \le p-1$. □

Definition 2.29. A periodic orbit $O^+(x)$ of period p is called an **attractor** (or **sink**) if for each $i \in \{0, 1, \ldots, p-1\}$, the point $f^i(x)$ is an attracting fixed point of f^p. Similarly, $O^+(x)$ is called a **repeller** (or **source**) if each $f^i(x)$ is a repelling fixed point of f^p.

Proposition 2.30. *Let $f : X \to X$ be continuous and $x \in X$. Then*

(i) $O^+(x)$ is an attractor if and only if $x = f^p(x)$ is an attracting fixed point of f^p;

(ii) $O^+(x)$ is a repeller if and only if $x = f^p(x)$ is a repelling fixed point of f^p.

Proof. Exercise. □

Remark 2.31. Similar to the fixed point case in Definition 2.26, a point y is called forward asymptotic to a periodic point x of period p if

$$\lim_{k \to \infty} f^{pk}(y) = x.$$

Example 2.32. For a fixed $b \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := b - x$. If x is a fixed point of f, then $x = b - x$, or $x = b/2$. Therefore, f has exactly one fixed point. However, observe that

$$f^2(x) = f(b - x) = b - (b - x) = x.$$

Therefore, every $x \in \mathbb{R}$ is periodic of period at most 2. None of the orbits are attracting, but they are all stable. The graph is plotted in Figure 2.6. □

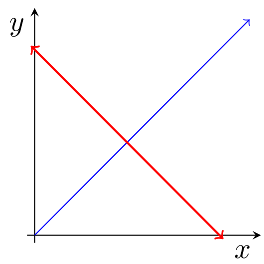


Figure 2.6: The function f is plotted in red together with the identity function in blue.

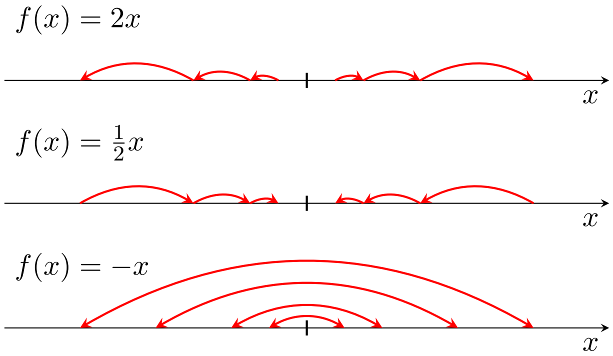
Chapter 3

One dimensional systems

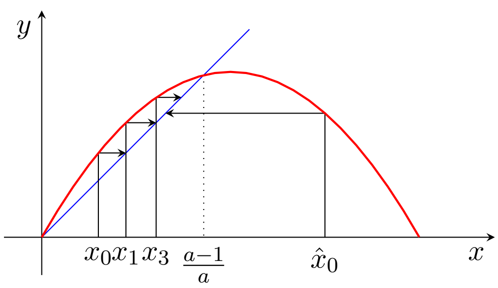
In this chapter, we look at dynamical systems defined by functions $f : I \to I$, where $I \subseteq \mathbb{R}$ is an interval. An advantage of studying one dimensional systems is that we can draw graphs.

Definition 3.1. A *phase portrait* of a dynamical system is a graphical representation of the possible paths of the system.

We use mainly use two techniques to visualize trajectories, and examples of them are given in the next figure. But be aware that graphs can be misleading!



(a) Different phase portraits.



(b) A plot of a function.

Figure 3.1

Example 3.2. We look at the recursion defined by $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = ax + b.$$

That is, $x_{n+1} = ax_n + b$. We have a number of different cases to consider based on the value of a. A plot can be seen in Figure 3.2.

Case 1: $a = 1$. So $x_{n+1} = x_n + b$, and the solution is $x_n = nb + x_0$. Now we consider the possible fixed points based on the value of b.

- If $b = 0$, then all points are fixed points.
- If $b > 0$, then there are no fixed points and $x_n \to \infty$ as $n \to \infty$.
- If $b < 0$, then there are no fixed points and $x_n \to -\infty$ as $n \to \infty$.

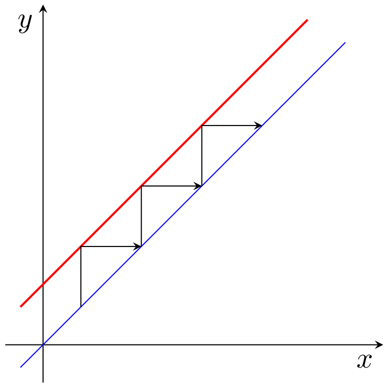


Figure 3.2: The plot of f when $a = 1$ and $b > 0$. Iterations are drawn in black and are seen to diverge to infinity.

Case 2: $a \neq 1$. In this case, f has exactly one fixed. The local behavior around the fixed point is similar to the exponential map, $x_{n+1} = ax_n$, which we will see shortly. The fixed point satisfies $x = ax + b$; therefore,

$$x = \frac{b}{1-a}$$

is the fixed point of f. We solve for the general solution to the recurrence:

$$\begin{aligned} x_1 &= ax_0 + b \\ x_2 &= a(ax_0 + b) + b = a^2 x_0 + ab + b \\ x_3 &= a(a^2 x_0 + ab + b) + b = a^3 x_0 + a^2 b + ab + b. \end{aligned}$$

We can see the general pattern emerging:

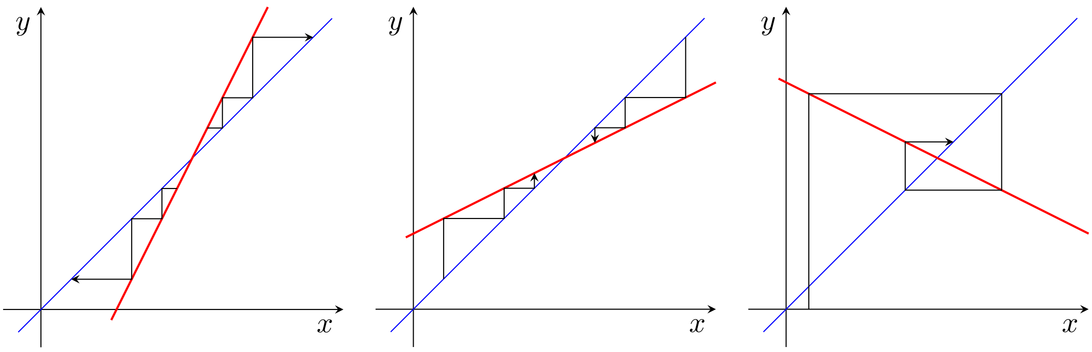
$$\begin{aligned} x_n &= a^n x_0 + b\left(\sum_{k=0}^{n-1} a^k\right) \\ &= a^n x_0 + b\frac{(1-a^n)}{(1-a)} \\ &= a^n\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}. \end{aligned} \tag{3.1}$$

Now we consider different values for a.

- If $a = -1$, then $x = \frac{b}{1-a}$ is a stable but not an attracting fixed point. All other points are periodic with minimal period 2 (see Example 2.32).
- If $|a| > 1$, then (a^n) diverges, so (x_n) diverges for $x_0 \neq \frac{b}{1-a}$. Thus $x = \frac{b}{1-a}$ is an unstable and repelling fixed point.
- If $|a| < 1$ then $\lim_{n\to\infty} a^n = 0$. Therefore, $x = \frac{b}{1-a}$ is a stable and attracting fixed point.

We want to further analyze the case when $|a| < 1$ as the sign of a changes the behavior of the system. However, we will only examine their phase diagrams in Figure 3.3. □

Remark 3.3. The linear difference equation we considered in Example 3.2 is essentially the only case where it is possible to write down an explicit expression for f^n, i.e. the solution (3.1). For a second degree polynomial, f^n is a polynomial of degree $2n$, which quickly grows out of hand.



(a) $1 < a$. (b) $0 < a < 1$. (c) $-1 < a < 0$.

Figure 3.3: Three phase diagrams for Example 3.2 with different values of a. The function f is plotted in red; the identity function is plotted in blue; and the iterations are plotted in black.

Definition 3.4. Let $f : I \to I$ be a continuously differentiable function. A periodic point x of minimal period p is called **hyperbolic** if $|(f^p)'(x)| \neq 1$. In particular, a *fixed point* x of a function f is hyperbolic if $|f'(x)| \neq 1$.

Recall the open ball of radius $r \geq 0$ about a point $x \in \mathbb{R}$ is

$$B_r(x) = \{y \in \mathbb{R} \mid \ |x - y| < r\}.$$

Theorem 3.5. *Let $f : I \to I$ be continuously differentiable, and suppose $x \in I$ is a fixed point of f.*

(i) If $|f'(x)| < 1$, then x is attracting and is thus a stable fixed point.

(ii) If $|f'(x)| > 1$, then x is repelling and is thus an unstable fixed point.

Proof. For (i), choose λ such that $|f'(x)| < \lambda < 1$. As f' is continuous, there exists $\delta > 0$ such that for all $y \in B_\delta(x)$, $|f'(y)| < \lambda$. Fix $y \in B_\delta(x)$ with $y \neq x$. By the Mean Value Theorem, there exists c in between y and x such that

$$\lambda > \left|f'(c)\right| = \frac{|f(x) - f(y)|}{|x - y|}.$$

Therefore,

$$|x - f(y)| = |f(x) - f(y)| < \lambda\, |x - y| < |x - y|.$$

This implies that $f(y) \in B_\delta(x)$, and so by the Mean Value Theorem, there exists c_1 between $f(y)$ and x such that

$$\lambda > \left|f'(c_1)\right| = \frac{|f(x) - f(f(y))|}{|x - f(y)|}.$$

Therefore,

$$\left|x - f^2(y)\right| < \lambda\, |x - f(y)| < \lambda^2\, |x - y| < |x - y|.$$

By induction, it follows that for all $n \in \mathbb{N}$,

$$|x - f^n(y)| < \lambda^n\, |x - y|.$$

Therefore $f^n(y) \to x$. Since this holds for all $y \in B_\delta(x)$, the fixed point x is attracting. By Proposition 2.24, x is also stable.

The proof for (ii) is similar to (i). Choose λ such that $|f'(x)| > \lambda > 1$, and since f' is continuous, there exists $\delta > 0$ such that for all $y \in B_\delta(x)$, $|f'(y)| > \lambda$. Again, let $y \in B_\delta(x)$ with $y \neq x$. Like above, by the Mean Value Theorem there is a c in between y and x such that

$$\lambda < \left|f'(c)\right| = \frac{|f(x) - f(y)|}{|x - y|}$$

and this time we have

$$|x - f(y)| = |f(x) - f(y)| > \lambda |x - y| > |x - y| .$$

If $\lambda |x - y| \geq \delta$, then we are done. Otherwise $f(y) \in B_\delta(x)$, and there exists c_1 in between x and $f(y)$ such that

$$\lambda < \left|f'(c_1)\right| = \frac{\left|f(x) - f^2(y)\right|}{|x - f(y)|}.$$

Thus,

$$\left|x - f^2(y)\right| = \left|f(x) - f^2(y)\right| > \lambda |x - f(y)| > \lambda^2 |x - y| .$$

We repeat this process as long as necessary; that is, until we find an m such that $\lambda^m \geq \delta/|x - y|$, and then

$$|x - f^m(y)| > \lambda^m |x - y| \geq \delta.$$

So that $f^m(y) \notin B_\delta(x)$. Since $\lambda > 1$, such an m exists. As $y \in B_\delta(x)$ was arbitrary, we see that x is a repelling and thus an unstable fixed point. □

An immediate consequence of Theorem 3.5 is that hyperbolic fixed points of a continuously differentiable system $f : \mathbb{R} \to \mathbb{R}$ are either attracting (and stable) or repelling (and unstable). The only possible uncertainty occurs when the derivative is exactly 1.

Example 3.6. Consider the logistic map $f_a(x) = ax(1 - x)$ for $1 < a < 2$. We have seen that the fixed points are $x = 0$ and $x = (a - 1)/a$. The derivative is $f_a'(x) = a - 2ax$, so

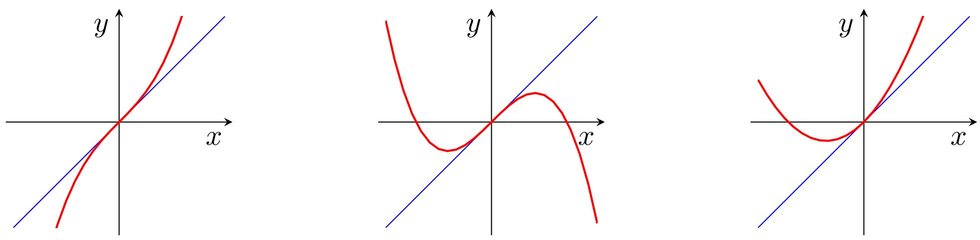
$$\left|f_a'(0)\right| = a > 1 \qquad \left|f_a'(a - 1)/a)\right| = |a - 2(a - 1)| = |2 - a| < 1.$$

Thus by Theorem 3.5, $x = 0$ is a repelling fixed point, and $x = (a - 1)/a$ is attracting.

Example 3.7. *What happens if $|f'(x)| = 1$?* In this case, there is no general answer. Consider the following functions, all with fixed point $x = 0$ and $|f'(0)| = 1$:

$$f_1(x) = x + x^3, \qquad f_2(x) = x - x^3, \qquad f_3(x) = x + x^2.$$

Their graphs are given in Figure 3.4.



(a) f_1 (b) f_2 (c) f_3

Figure 3.4: The fixed point $x = 0$ is (a) a repeller, (b) an attractor, and (c) an unstable point that is neither repeller or attractor.

Proposition 3.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with a fixed point at $x = 0$.*

(i) If there exists $r > 0$ such that for all $x \in B_r(0) \setminus \{0\}$, $|f(x)| < |x|$, then $x = 0$ is an attracting fixed point.

(ii) If there exists $r > 0$ such that for all $x \in B_r(0) \setminus \{0\}$, $|f(x)| > |x|$, then $x = 0$ is a repelling fixed point.

Proof. First we fix some notation for the proof. For some $x_0 \in \mathbb{R}$ and for all $n \in \mathbb{N}$, let $x_n = f(x_{n-1})$.

For (i), suppose that for all $x \in B_r(0) \setminus \{0\}$, $|f(x)| < |x|$. Then $x \in B_r(0)$ implies that $|f(x)| \leq |x| < r$, so $f(x) \in B_r(0)$. Because $|f(x)| < |x|$ for all $x \in B_r(0) \setminus \{0\}$, it follows that f has no other fixed points contained in $B_r(0)$, so $|f(x)| = |x|$ if and only if $x = 0$. Let $x_0 \in B_r(0) \setminus \{0\}$. For all $n \in \mathbb{N}$,

$$0 \leq |x_n| \leq |x_{n-1}| < r, \tag{3.2}$$

so that the sequence $(|x_n|)$ is monotonically decreasing and bounded from below. Thus, the sequence converges. Let

$$\bar{x} = \lim_{n\to\infty} |x_n|\,.$$

If there exists some $k \in \mathbb{N}$ such that $x_k = 0$, then $\bar{x} = 0$, and we are done. Otherwise, we assume that for all $k \in \mathbb{N}$, $|x_k| > 0$. Then there exists a subsequence (x_{n_k}) of (x_n) such that all the x_{n_k} have the same sign. Then from (x_{n_k}), choose a another subsequence $(x_{n_{k_j}})$ such that all the $f(x_{n_{k_j}})$ have the same sign. Then

$$\lim_{n\to\infty} x_{n_{k_j}} = \bar{x}.$$

For all j, $|x_{n_{k_j}}| > |f(x_{n_{k_j}})| = |x_{n_{k_j}+1}|$. When we take the limit,

$$|\bar{x}| \geq \lim_{j\to\infty} \left| f(x_{n_{k_j}}) \right| = |\bar{x}|\,.$$

Thus, $|\bar{x}| = \lim_{j\to\infty} |f(x_{n_{k_j}})|$. Since f is continuous, $|\bar{x}| = |f(\bar{x})|$. Because $\bar{x} \in B_r(0)$, it follows that $\bar{x} = 0$.

For (ii), suppose for all $x \in B_r(0) \setminus \{0\}$, $|f(x)| > |x|$, and let $x_0 \in B_r(0) \setminus \{0\}$. Again as before, f cannot have a fixed point in $B_r(0) \setminus \{0\}$. Then $|x_1| > |x_0|$, and if $|x_1| \geq r$ then we are done. Otherwise, $|x_2| > |x_1|$ and so on. Either $|x_n| < r$ for all n, or for some k, $|x_k| \geq r$. In the latter case, we are done. In the former case, $(|x_n|)$ is a monotonically increasing sequence with $0 < |x_n| < |x_{n+1}| < r$ for all n. Therefore, $\lim_{n\to\infty} |x_n| := \bar{x}$ exists. As above, by choosing subsequences we see that we must have $\bar{x} = f(\bar{x})$. As $\bar{x} > 0$, the only remaining possibility is that $\bar{x} = r$. Then choosing $\delta = r/2$, we see that 0 is a repelling fixed point. □

The attracting fixed point, $x = 0$, in Proposition 3.8 is a stable fixed point by Proposition 2.24 since we assumed the function to be one-dimensional and continuous.

Example 3.9. We can not remove the continuity condition in Proposition 3.8. To see this, consider first the function $f : [0, 1) \to [0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2^{k+1}} & \text{for } 1/2^k < x \leq 1/2^{k-1}, \\ 0 & \text{for } x = 0. \end{cases}$$

By definition, $x = 0$ is a fixed point. Moreover, $f(x) < x$ for all $x \in (0, 1)$, so the forward orbit $O^+(x)$ of any point x is a decreasing sequence. But for any $x \in (0, 1)$, we can find a k such that $1/2^k < x \leq 1/2^{k-1}$, and for any such x and k we have

$$\lim_{n\to\infty} f^n(x) = \frac{1}{2^k} \neq 0.$$

Thus, the fixed point $x = 0$ is not attracting. It is, however, stable. (Can you prove this?) The plot of this can be seen in Figure 3.5. □

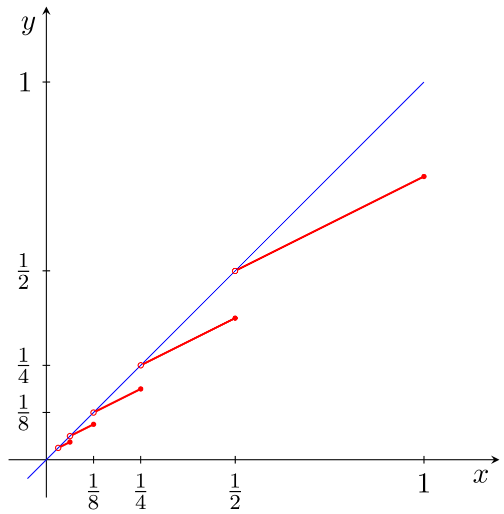


Figure 3.5: The plot of Example 3.9.

Example 3.10. Now let us consider the similar function to the one in Example 3.9:

$$g(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2^k} & \text{for } 1/2^k \le x < 1/2^{k-1}, \\ 0 & \text{for } x = 0. \end{cases}$$

By definition, $x = 0$ is a fixed point. Moreover, $g(x) > x$ for all $x \in (0, 1)$, so the forward orbit $O^+(x)$ of any such point x is an increasing sequence. For every $r > 0$, there exists $x \in (0, 1)$ and $k \ge 1$ such that $1/2^k < x \le 1/2^{k-1} < r$. Then

$$\lim_{n\to\infty} g^n(x) = \frac{1}{2^{k-1}} < r,$$

Hence, the orbit never leaves the ball of radius of r, so that the fixed point $x = 0$ is not repelling. The plot of this function is given in Figure 3.6. □

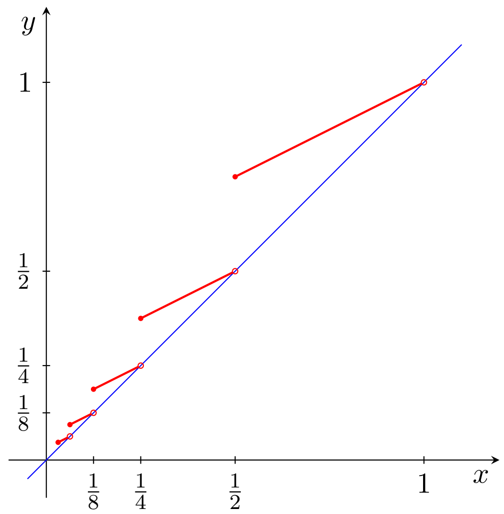


Figure 3.6: The plot of Example 3.10.

3.1 Periodic points and Sharkovsky's Theorem

In Example 2.32, the dynamical system defined by a function of the form $f : \mathbb{R} \to \mathbb{R}$, $f(x) = b - x$ has one fixed point at $x = \frac{b}{2}$. By calculating $f^2(x)$, we saw that the orbits of all other points are periodic of period 2. In general however, attempting to find periodic orbits by calculating f^n is not so simple.

Example 3.11. Let $f(x) = x^2 - 1$. Then f has two fixed points: such a fixed point satisfies $x^2 - x - 1 = 0$, so

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Does the function f have some periodic points (with period $p > 1$)? First, we compute a formula for f^2:

$$f^2(x) = (x^2 - 1)^2 - 1 = x^4 - 2x^2.$$

The periodic points of period two are the fixed points of f^2, and such points must satisfy $x = x^4 - 2x^2$ or, equivalently,

$$x(x^3 - 2x - 1) = 0. \tag{3.3}$$

All fixed points are also periodic of period two, so they must also satisfy the above quartic equation (3.3). Since $x = 0$ is not a fixed point, we must have that $x^2 - x - 1$ divides the cubic factor in (3.3). Therefore,

$$0 = x(x^3 - 2x - 1) = x(x+1)\left(2x - 1 + \sqrt{5}\right)\left(2x - 1 - \sqrt{5}\right).$$

Thus, the two periodic points of minimum period 2 are $x = 0$ and $x = -1$.

We could try to replicate this process used in Example 3.11 for f^3 and so on for higher n, but it would quickly become impractical. Thankfully, we have some other tools at our disposal.

The next theorem is due to Sharkovsky who proved it in 1964. In order to state the next theorem, we need to define a total order on $\mathbb{N}$. Define **Sharkovsky's order** $\prec$ with the following ascending sequence:

$$\begin{array}{ccccccccccc}
 & & 3 & \prec & 5 & \prec & 7 & \prec & 9 & \prec & \cdots \\
\cdots & \prec & 2\cdot 3 & \prec & 2\cdot 5 & \prec & 2\cdot 7 & \prec & 2\cdot 9 & \prec & \cdots \\
\cdots & \prec & 2^2\cdot 3 & \prec & 2^2\cdot 5 & \prec & 2^2\cdot 7 & \prec & 2^2\cdot 9 & \prec & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\cdots & \prec & 2^n\cdot 3 & \prec & 2^n\cdot 5 & \prec & 2^n\cdot 7 & \prec & 2^n\cdot 9 & \prec & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\cdots & \prec & 2^4 & \prec & 2^3 & \prec & 2^2 & \prec & 2 & \prec & 1.
\end{array}$$

So that 3 is the smallest integer and 1 the largest integer relative to Sharkovsky's order $\prec$. And more generally, every odd integer n has only finitely many integers k such that $k \prec n$, and every even integer n has only finitely many integers k such that $n \prec k$.

Theorem 3.12 (Sharkovsky's Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a periodic point of minimal period p and $p \prec q$, then f has a periodic point of minimal period q.*

The following is an important corollary, and follows from the definition of Sharkovsky's order

Corollary 3.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a periodic point of minimal period 3, then it has a periodic point of minimal period k for all $k \in \mathbb{N}$.*

Example 3.14. We will create a function that has a periodic point with minimal period $p \geq 1$ for all $p \geq 1$. By Sharkovsky's Theorem, it is enough to create a continuous function with a periodic point of minimum period 3.

Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree 2 such that $f(0) = 1$, $f(1) = 2$ and $f(2) = 0$. Then $f(x) = (x-2)(ax+b)$ for some $a, b \in \mathbb{R}$, so

$$1 = f(0) = -2b, \qquad\qquad 2 = f(1) = -(a+b).$$

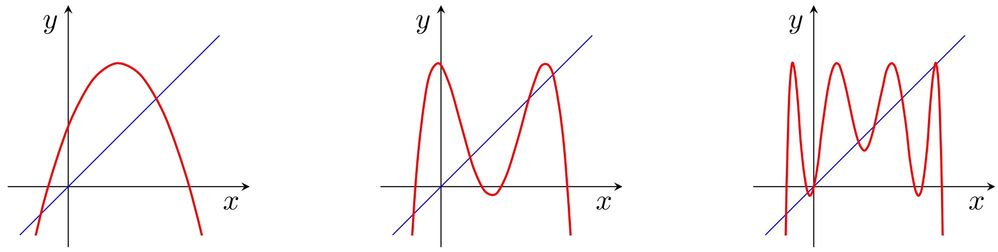
Thus, $a = -3/2$ and $b = -1/2$, and so

$$f(x) = -\frac{1}{2}(x-2)\,(3x+1)\,.$$

One checks that

$$f^2(x) = -(27x^4 - 90x^3 + 69x^2 + 10x - 16)/8,$$

and f^3 is then a polynomial of degree 8. These graphs can be seen in Figure 3.7. □



(a) f. (b) f^2. (c) f^3.

Figure 3.7: The graphs of f, f^2, and f^3 from Example 3.14.

We prove a special case of Sharkovsky's Theorem.

Proposition 3.15. *Let $f : I \to I$ be a continuous function. If f has a periodic point of minimal period 2 in I then f has a fixed point in I.*

Proof. Let x_0 be a periodic point of f with minimal period 2, so that we have $x_1 = f(x_0)$ and $x_0 = f(x_1)$. Without loss of generality we may assume $x_0 < x_1$ (since both x_0 and x_1 have minimal period 2). Thus,

$$x_0 < f(x_0), \tag{3.4}$$
$$x_1 > f(x_1). \tag{3.5}$$

From (3.4), f must lie above the identity function at x_0, and from (3.5), f must lie below the identity function at x_1. Since f is continuous, by the Intermediate Value Theorem there must be an $\bar{x} \in (x_0, x_1)$ such that $\bar{x} = f(\bar{x})$. Hence, f has a fixed point. □

Proposition 3.15 gives us a little bit of milage. Of course, this is also a special case of Sharkovsky's Theorem.

Corollary 3.16. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a periodic point of minimal period 2^n, then f has a periodic point of minimal period 2^k for every k such that $0 \leq k \leq n$.*

Proof. Consider the function $h_{n-1} := f^{2^{n-1}}$. As f is continuous, h_{n-1} is also continuous. Since f has a periodic point x of minimal period 2^n,

$$h_{n-1}^2(x) = \left(f^{2^{n-1}}\right)^2(x) = f^{2^n}(x) = x.$$

That is, h_{n-1} has a periodic point of minimal period 2, and therefore by Proposition 3.15, it must have a fixed point. By the definition of h_{n-1}, it follows that f has a periodic point of minimal period 2^{n-1}. Repeating this argument gives the result. □

Chapter 4

Bifurcations in one dimensional systems

Consider a family of maps $f_a : \mathbb{R} \to \mathbb{R}$, defining a family of discrete dynamical systems depending on a parameter $a \in \mathbb{R}$. We have already seen an example of this: the logistic map in Section 1.1.3. It has fixed points $x = 0$ and $x = (a-1)/a$ (for $a > 0$). If $a \neq 1$, the fixed points are distinct, but if $a = 1$, there is only one fixed point. Further, for $0 < a < 1$, the fixed point $x = 0$ is attracting, for $a > 1$ it is repelling, and for $a = 1$ it is neither attracting nor repelling. This sudden change of behavior is the topic of this section.

When a small, smooth change to a results in a dramatic change in the qualitative behavior of the system, we have a **bifurcation**. The logistic map has a bifurcation at $a = 1$. We shall come back to this example when talking about transcritical bifurcations in Section 4.2.

Example 4.1. Consider the family of functions $f_b(x) = -x^2 + x + b$. The fixed points are $x = \pm\sqrt{b}$, when possible. We will consider three cases, and we plot f_b for different values of b in Figure 4.1.

Case 1: $b < 0$. The are no (real) fixed points in this system.

Case 2: $b = 0$. There is exactly one fixed point: $x = 0$. However, since $f_b'(x) = 1 - 2x$, it follows that $f_b'(0) = 1$, which is no help to us. But the map in this case is exactly the logistic map with $a = 1$, so we know that the fixed point $x = 0$ is neither attracting nor repelling.

Case 3: $b > 0$. There are two distinct fixed points. For $x = \sqrt{b}$,

$$\left| f_b'\left(\sqrt{b}\right) \right| = \left|1 - 2\sqrt{b}\right|.$$

Thus, by Theorem 3.5, when $0 < b < 1$, then $x = \sqrt{b}$ is an attracting stable fixed point. But when $b > 1$, then $x = \sqrt{b}$ is a repelling unstable fixed point. Note that Theorem 3.5 is inconclusive when $b = 1$. On the other hand, for $x = -\sqrt{b}$,

$$\left| f_b'\left(-\sqrt{b}\right) \right| = \left|1 + 2\sqrt{b}\right| > 1$$

for all $b > 0$. Therefore, $x = -\sqrt{b}$ is always a repelling unstable fixed point. □

Another useful tool to analyze the behavior of 1-parameter family of functions is to plot the **bifurcation diagram**[1]. That is, a plot of the fixed points x as a function of the parameter b. In this example, that is the solution to $x = f_b(x)$: namely, $x^2 - b = 0$. This is plotted in Figure 4.2; to distinguish between stable and unstable, normally one uses solid and dashed lines,

[1] One of the most famous bifurcation diagrams is the one from the logistic map.

respectively. Here, the bifurcation diagram shows how we have two fixed points for positive b, which remain as b decreases towards zero, when suddenly, we have one fixed point and then no fixed points. □

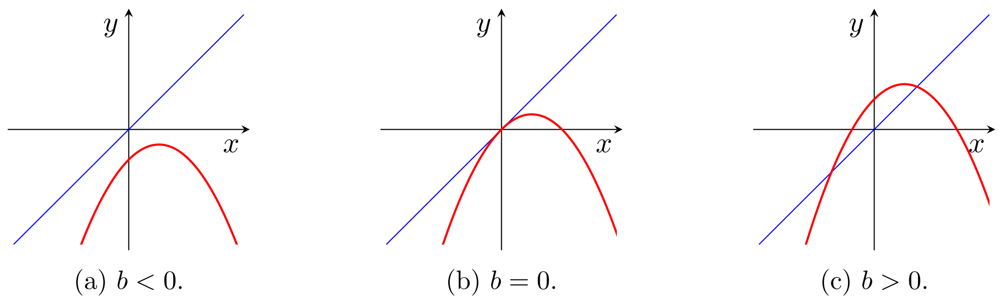


Figure 4.1: Plots of the family of functions f_b from Example 4.1 for three sets of values for b.

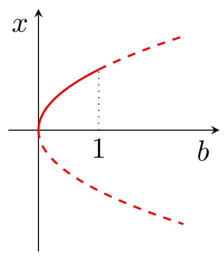


Figure 4.2: The bifurcation diagram from Example 4.1, plotting $x^2 = b$. The solid line signifies stability, whereas the dashed line signifies instability.

A major ingredient when studying bifurcations is the Implicit Function Theorem (IFT) from analysis. The IFT allows us to convert implicit expressions into explicit ones. That is, it gives conditions under which an equation of the form $G(x, y) = 0$ is equivalent—at least locally—to an expression of the form $y = h(x)$.

Theorem 4.2 (Implicit Function Theorem)**.** *Let $G : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable on an open set $U \subseteq \mathbb{R}^2$. If for $(x_0, y_0) \in U$, $G(x_0, y_0) = 0$ and*

$$\frac{\partial G}{\partial y}(x_0, y_0) \neq 0,$$

then there exist open intervals $I, J \subseteq \mathbb{R}$, with $x_0 \in I$, $y_0 \in J$, and $I \times J \subseteq U$, and a continuously differentiable function $h : I \to J$ such that $h(x_0) = y_0$ and for all $x \in I$, $G(x, h(x)) = 0$.

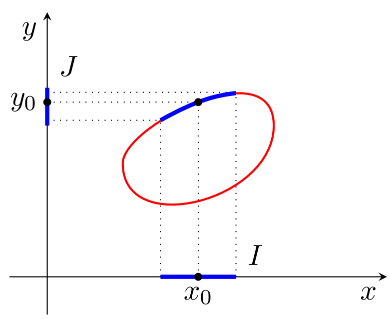


Figure 4.3: An example of the Implicit Function Theorem. Here, $G(x, y) = 0$ is plotted in red, with the function $h(x)$ in blue. The intervals I and J are also filled in with blue on the axes.

We plot an example of Implicit Function Theorem in Figure 4.3. Note that in a sufficiently small interval I, the function h is unique. Furthermore, if G is r times continuously differentiable, then so is h as

$$h'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}.$$

The IFT allows us to locally express fixed points x of a function f_a in terms of the parameter a; that is, it gives us conditions for the existence of an expression $x = h(a)$. To apply the theorem, we need to be a little more precise: rather than $f_a : I \to I$, $a \in J$, we write $f_a(x) = F(a, x)$, where $F : J \times I \to I$. Then fixed points of f_a are the $x \in I$ such that for some $a \in J$, $F(a, x) = x$.

Corollary 4.3. *Let $I, J \subset \mathbb{R}$ and $F : J \times I \to I$ be a continuously differentiable function. If for some $\bar{a} \in J$, $\bar{x} \in I$, $F(\bar{a}, \bar{x}) = \bar{x}$ and*

$$\frac{\partial F}{\partial x}(\bar{a}, \bar{x}) \neq 1,$$

then there exist open intervals $K \subseteq I$, $L \subseteq J$ with $\bar{x} \in K$, $\bar{a} \in L$, and a unique function $h : L \to K$ such that $h(\bar{a}) = \bar{x}$ and for all $a \in K$, $F(a, h(a)) = h(a)$.

Proof. Let $G(a, x) := F(a, x) - x$. Then $G(a, x) = 0$ if and only if $F(a, x) = x$, and $\frac{\partial G}{\partial x}(a, x) \neq 0$ if and only if $\frac{\partial F}{\partial x}(a, x) \neq 1$. Then we apply the IFT to G. $\square$

Recall that a fixed point x of a function f_a is hyperbolic if $|f_a'(x)| \neq 1$. Corollary 4.3 shows us that around any hyperbolic fixed point there is a neighborhood within which we may vary the parameter a without changing the properties of the fixed point and without any new fixed points emerging. In the cases where a fixed point is *not* hyperbolic, we may see some more interesting behavior—namely, a bifurcation!

4.1 Saddle node bifurcation

A bifurcation like the one in Example 4.1, where as the parameter varies, two fixed points approach each other, then coincide and disappear, is called a **saddle-node bifurcation** . This is the simplest scenario where two fixed points "collide" and mutually annihilate each other. The next theorem helps us analyze the behavior of systems with a non-hyperbolic fixed point that is also a saddle-node.

Recall the definition of the **differentiability classes**. The class of continuous functions is denoted by C^0, and we define C^k recursively for positive integers k; namely, C^k is the class of functions whose (partial) derivatives lie in C^{k-1}. So a function is in C^1 if it is continuously differentiable, and a function is in C^2 if its derivative is continuously differentiable. A function is **smooth** if it is contained in C^∞.

Theorem 4.4 (Saddle-node bifurcation)**.** *Let $I, J \subset \mathbb{R}$ and $F : J \times I \to I$ be a C^2-function. Let $f_a(x) = F(a, x)$ for all $a \in J, x \in I$. Suppose that there is a point $(\bar{a}, \bar{x}) \in J \times I$ such that*

1. $f_{\bar{a}}(\bar{x}) = F(\bar{a}, \bar{x}) = \bar{x}$,
2. $f_{\bar{a}}'(\bar{x}) = \frac{\partial F}{\partial x}(\bar{a}, \bar{x}) = 1$,
3. $f_{\bar{a}}''(\bar{x}) = \frac{\partial^2 F}{\partial x^2}(\bar{a}, \bar{x}) \neq 0$, *and*
4. $\frac{\partial F}{\partial a}(\bar{a}, \bar{x}) \neq 0$.

Then there exists an open interval K, containing $\bar{x}$, and a C^2-function $h : K \to \mathbb{R}$ such that $h(\bar{x}) = \bar{a}$ and $f_{h(x)}(x) = x$. Moreover, $h'(\bar{x}) = 0$ and $h''(\bar{x}) \neq 0$.

Proof. Consider the function $G(a,x) = F(a,x) - x$. Then by (1) $G(\bar{a},\bar{x}) = 0$, and by (4)

$$\frac{\partial G}{\partial a}(\bar{a},\bar{x}) \neq 0.$$

By the Implicit Function Theorem, there exists an interval K and a function $h : K \to \mathbb{R}$ such that $h(\bar{x}) = \bar{a}$ and for all $x \in K$, $G(h(x),x) = 0$. That is, $f_{h(x)}(x) = x$.

Let us turn to the statement concerning h' and h''. We implicitly differentiate[2] the expression $G(h(x),x) = 0$ with respect to x:

$$0 = \frac{d}{dx}(G(h(x),x)) = \frac{\partial G}{\partial a}(h(x),x) \cdot h'(x) + \frac{\partial G}{\partial x}(h(x),x)\,. \tag{4.1}$$

Solving for h' and setting $x = \bar{x}$ gives

$$h'(\bar{x}) = -\frac{\frac{\partial G}{\partial x}(\bar{a},\bar{x})}{\frac{\partial G}{\partial a}(\bar{a},\bar{x})} = -\frac{\frac{\partial F}{\partial x}(\bar{a},\bar{x}) - 1}{\frac{\partial F}{\partial a}(\bar{a},\bar{x})} = -\frac{0}{\frac{\partial F}{\partial x}(\bar{a},\bar{x})} = 0$$

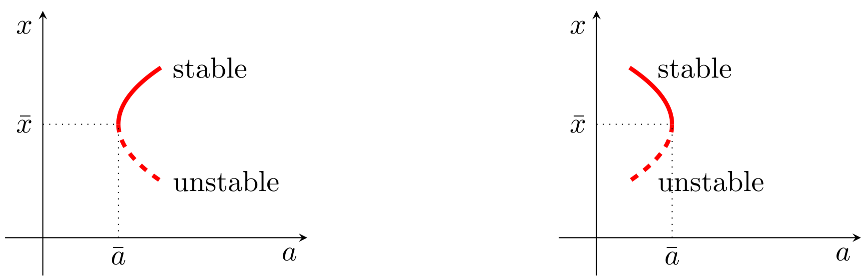
Implicitly differentiating (4.1) with respect to x, it follows that (can you show this?)

$$h''(\bar{x}) = -\frac{\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})}{\frac{\partial F}{\partial a}(\bar{a},\bar{x})}. \tag{4.2}$$

By (3), $h''(\bar{x}) \neq 0$, and we are done. □

If the conditions of Theorem 4.4 are satisfied, the point $(\bar{a},\bar{x})$ is a saddle-node bifurcation. The theorem allows us to examine stability around the fixed point by looking at the second derivative of $f_{h(x)}(x)$. By assumption we have $f'_{\bar{a}}(\bar{x}) = 1$ and $f''_{\bar{a}}(\bar{x}) \neq 0$, so that $f''_{h(x)}(x) \neq 0$ in a neighborhood of $\bar{x}$. From Equation (4.2), if $h''(x) > 0$, then $f''_{\bar{a}}(\bar{x})$ and $\frac{\partial F}{\partial a}(\bar{a},\bar{x})$ have opposite signs, and if $h''(x) < 0$, then they have the same sign. Let's consider two sets of values for $f''_{\bar{a}}(\bar{x})$.

Case 1: $f''_{\bar{a}}(\bar{x}) < 0$. If $x < \bar{x}$, it follows that $f'_{h(x)}(x) > 1$, and if $x > \bar{x}$, then $f'_{h(x)}(x) < 1$. Plots of these situations are seen in Figure 4.4.



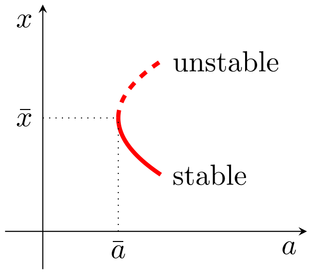
(a) $h''(x) > 0$.

(b) $h''(x) < 0$.

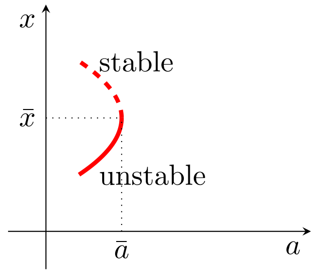
Figure 4.4: Plots of $a = h(x)$.

Case 2: $f''_{\bar{a}}(\bar{x}) > 0$. Similar to above, if $x < \bar{x}$, then $f'_{h(x)}(x) < 1$, and when $x > \bar{x}$, then $f'_{h(x)}(x) > 1$. Again, we plot these bifurcation diagrams in Figure 4.5.

[2]Recall the chain rule.



(a) $h''(x) > 0$.



(b) $h''(x) < 0$.

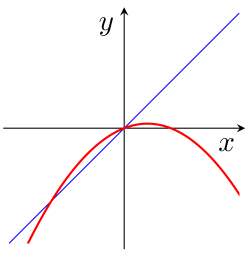
Figure 4.5: Plots of $a = h(x)$.

4.2 Transcritical bifurcation

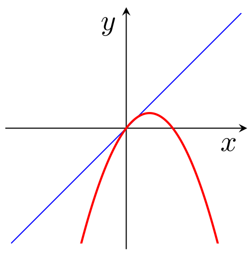
In some situations, there are fixed points that should never disappear like in the saddle-node bifurcation. For example, in the logistic model for population, $x = 0$ is a fixed point that should remain a fixed point regardless of the value of a. However, it *stability* can change, and bifurcations such as these are called **transcritical bifurcations**.

Example 4.5. Let $f_a(x) := ax(1-x)$. The fixed points of f_a are $x = 0$ and $x = (a-1)/a$, provided $a \neq 0$. Set $F(a,x) = f_a(x)$. There is a bifurcation at $(\bar{a},\bar{x}) = (1,0)$, but this is not a saddle node bifurcation since $\frac{\partial F}{\partial a}(1,0) = 0 \neq 1$. For the stability, we recall Theorem 3.5, so we consider the derivative: $f_a'(x) = a(1-2x)$. We look at the two fixed points.

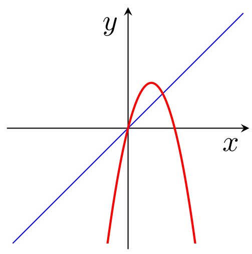
- For $x = 0$, $|f_a'(0)| = |a|$. Therefore, $x = 0$ is stable when $|a| < 1$ and unstable when $|a| > 1$.
- For $x = (a-1)/a$, $|f_a'(x)| = |2-a|$. Thus, x is a stable when $1 < a < 3$ and unstable when $|a| < 1$ or $|a| > 3$.



(a) $a < 1$.



(b) $a = 1$.



(c) $a > 1$.

Figure 4.6: We plot the function $f_a(x) = ax(1-x)$ for different values of a in red together with the identity function in blue.

There are two fixed points for all a, except $a = 1$, and the stability properties of the fixed points change at $a = 1$. The bifurcation diagram is plotted in Figure 4.7. $\square$

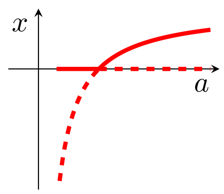


Figure 4.7: The bifurcation diagram for Example 4.5. We plot two curves $x = 0$ and $x = (a-1)/a$. This shows an example of a transcritical bifurcation.

In the proof of the next theorem, we use a Taylor expansion to compute a partial derivative. Recall that the Taylor expansion of a function $\psi : \mathbb{R} \to \mathbb{R}$ about a point $\bar{x} \in \mathbb{R}$ is

$$\psi(x) = \psi(\bar{x}) + \psi'(\bar{x})(x-\bar{x}) + \frac{1}{2}\psi''(\bar{x})(x-\bar{x})^2 + O\left((x-\bar{x})^3\right).$$

All the higher order terms in $O\left((x-\bar{x})^3\right)$ are also multiples of $(x-\bar{x})^3$.

Theorem 4.6 (Transcritical bifurcation (special case))**.** *Let $I, J \subset \mathbb{R}$ and $F : J \times I \to I$ be a C^2-function. Suppose that there is a point $(\bar{a}, \bar{x}) \in J \times I$ such that*

1. *for all $a \in J$, $F(a, \bar{x}) = \bar{x}$,*
2. $\frac{\partial F}{\partial x}(\bar{a}, \bar{x}) = 1$,
3. $\frac{\partial^2 F}{\partial x^2}(\bar{a}, \bar{x}) \neq 0$, *and*
4. $\frac{\partial^2 F}{\partial a \partial x}(\bar{a}, \bar{x}) \neq 0$.

Then there exists an open interval $K \subseteq J$ and a unique C^1-function $g : K \to I$ such that $g(\bar{a}) = \bar{x}$ and for all $a \in K \setminus \{\bar{a}\}$, $g(a) \neq \bar{x}$ and $F(a, g(a)) = g(a)$.

Proof. Define $G(a,x) = F(a,x) - x$. We cannot apply the IFT as in the previous theorems since both

$$\frac{\partial G}{\partial a}(\bar{a}, \bar{x}) = 0, \qquad \frac{\partial G}{\partial x}(\bar{a}, \bar{x}) = 0.$$

We will define a new function, but before we do that, note that (1) implies that

$$\begin{aligned}\lim_{x\to\bar{x}} \frac{F(a,x)-x}{x-\bar{x}} &= \lim_{x\to\bar{x}} \frac{F(a,x)-\bar{x}-(x-\bar{x})}{x-\bar{x}} \\ &= \lim_{x\to\bar{x}} \frac{F(a,x)-F(a,\bar{x})}{x-\bar{x}} - 1 \\ &= \frac{\partial F}{\partial x}(a, \bar{x}) - 1.\end{aligned} \tag{4.3}$$

Now, define

$$H(a,x) := \begin{cases} \dfrac{F(a,x)-x}{x-\bar{x}} & x \neq \bar{x}, \\ \dfrac{\partial F}{\partial x}(a,\bar{x}) - 1 & x = \bar{x}. \end{cases}$$

By (4.3), since F is a C^2-function, $H(a,x)$ is a C^1-function. We will show that H satisfies the other conditions of the Implicit Function Theorem. By assumption (4) and (4.3), $H(\bar{a}, \bar{x}) = 0$, so we must also show that $\frac{\partial H}{\partial x}(\bar{a}, \bar{x}) \neq 0$.

Now we need to apply our Taylor expansion to $\psi(x) = F(a,x) - x$. From assumption (1), $\psi(\bar{x}) = 0$, so the constant term vanishes. Thus for a fixed a,

$$F(a,x) - x = \left(\frac{\partial F}{\partial x}(a,\bar{x}) - 1\right)(x-\bar{x}) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(a,\bar{x})\right)(x-\bar{x})^2 + O((x-\bar{x})^3). \tag{4.4}$$

From the definition of H and applying (4.4),

$$\begin{aligned}\frac{\partial H}{\partial x}(a,x) &= \frac{\partial}{\partial x}\left(\frac{F(a,x)-x}{x-\bar{x}}\right) \\ &= \frac{\left(\frac{\partial F}{\partial x}(a,x)-1\right)(x-\bar{x}) - (F(a,x)-x)}{(x-\bar{x})^2} \\ &= \frac{\left(\frac{\partial F}{\partial x}(a,x)-1\right)(x-\bar{x}) - \left(\frac{\partial F}{\partial x}(a,\bar{x})-1\right)(x-\bar{x})}{(x-\bar{x})^2} - \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a,\bar{x}) + O(x-\bar{x}) \\ &= \frac{\frac{\partial F}{\partial x}(a,x) - \frac{\partial F}{\partial x}(a,\bar{x})}{x-\bar{x}} - \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a,\bar{x}) + O(x-\bar{x})\end{aligned}$$

Now to finish our Herculean effort, we take the limit as $x \to \bar{x}$. Since H is C^1,

$$\frac{\partial H}{\partial x}(a,\bar{x}) = \lim_{x\to\bar{x}} \frac{\partial H}{\partial x}(a,x) = \frac{\partial^2 F}{\partial x^2}(a,\bar{x}) - \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a,\bar{x}) = \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a,\bar{x}).$$

By assumption (3), $\frac{\partial^2 F}{\partial x^2}(a,\bar{x}) \neq 0$, so $\frac{\partial H}{\partial x}(\bar{a},\bar{x}) \neq 0$.

Now we have shown that the Implicit Function Theorem applies to H. Thus, there exists a unique function $g : K \to I$ such that $g(\bar{a}) = \bar{x}$ and $H(a, g(a)) = 0$, which gives $F(a, g(a)) = g(a)$ for all $a \in K$. By implicitly differentiating $H(a, g(a)) = 0$ with respect to a,

$$g'(\bar{a}) = -\frac{\frac{\partial H}{\partial a}(\bar{a},\bar{x})}{\frac{\partial H}{\partial x}(\bar{a},\bar{x})} = (-2)\frac{\frac{\partial}{\partial a}\left(\frac{\partial F}{\partial x}(\bar{a},\bar{x}) - 1\right)}{\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})} = (-2)\frac{\frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x})}{\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})} \neq 0. \tag{4.5}$$

This shows that $g(a) \neq \bar{x}$ in some interval around $\bar{a}$. $\square$

We analyze the implications of Theorem 4.6. We will see, mathematically, what we saw in Example 4.5; namely, the number of fixed points remains constant, but their stability changes. We have two cases to consider: (1) $x = \bar{x}$, the fixed point of $f_a(x)$ and (2) $x = g(a)$, the function produced by Theorem 4.6.

Observe that the first two assumptions of Theorem 4.6 require that the system $f_a(x)$ have a bifurcation at $(\bar{a},\bar{x})$ since the first requires $\bar{x}$ is a fixed point and the second requires $f'_{\bar{a}}(\bar{x}) = 1$.

Case 1: $x = \bar{x}$. Because

$$\frac{\partial F}{\partial x}(\bar{a},\bar{x}) = 1 \qquad\qquad \frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x}) \neq 0,$$

the first partial $\frac{\partial F}{\partial x}$ is changing with respect to a. Therefore, (think back to Theorem 3.5) the stability must change right at $a = \bar{a}$. In other words for $a < \bar{a} < b$, the stability at $(a,\bar{x})$ is different from the stability at $(b,\bar{x})$.

Case 2: $x = g(a)$. Let us consider the Taylor expansion of $\frac{\partial F}{\partial x}$ around $\bar{a}$; recall that $g(\bar{a}) = \bar{x}$. We apply the formula (4.5) to get

$$\begin{aligned}
\frac{\partial F}{\partial x}(a,g(a)) &= \frac{\partial F}{\partial x}(\bar{a},\bar{x}) + \left(\frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x}) + \frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})\frac{dg}{da}(\bar{a})\right)(a-\bar{a}) + O\left((a-\bar{a})^2\right)\\
&= 1 + \left(\frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x}) - (2)\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})\frac{\frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x})}{\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})}\right)(a-\bar{a}) + O\left((a-\bar{a})^2\right)\\
&= 1 - \frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x})(a-\bar{a}) + O\left((a-\bar{a})^2\right).
\end{aligned}$$

The relevant approximation of values a near $\bar{a}$ is

$$\frac{\partial F}{\partial x}(a,g(a)) \approx 1 - \frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x})(a-\bar{a}).$$

the stability changes at $a = \bar{a}$, but the change is in the opposite direction to the change for $x = 0$.

In summary, with a transcritical bifurcation, we have an *exchange of stabilities* between two fixed points; see another example in Figure 4.8.

(a) $\frac{\partial^2 F}{\partial a \partial x}(\bar{a},\bar{x}) > 0.$ (b) $\frac{\partial^2 F}{\partial a \partial x}(\bar{a},\bar{x}) < 0.$

Figure 4.8: Two examples of transcritical bifurcations.

4.3 Pitchfork bifurcation

The third kind of bifurcation is the **pitchfork bifurcation**, which involves systems with some kind of *symmetry*. For example, a spacial symmetry between left and right. This description may not be clear in the examples we consider, but the *name* should be clear in the bifurcation diagrams. Before we explain further, we consider an example.

Example 4.7. Let $f_a(x) = x + ax - x^3$ The fixed points of f_a satisfy $x = x + ax - x^3$. Therefore, the potential fixed points are $x = 0$ and $x = \pm\sqrt{a}$. We consider the fixed points for different values of a. Figure 4.9 plots the three cases we consider.

Case 1: $a < 0$. We have only one fixed point; namely, $x = 0$. Thus,

$$\left|f_a'(0)\right| = |1 + a|,$$

When $-2 < a < 0$, then $x = 0$ is a stable and attracting fixed point, and for $a < -2$, $x = 0$ is unstable and repelling.

Case 2: $a = 0$. Like in the first case, we still have only one fixed point: $x = 0$. However, $f_a'(0) = 1$, so its stability is not immediately clear. Consider instead

$$|f_0(x)| = |x - x^3| = |x(1 - x^2)| = |x||1 - x^2|.$$

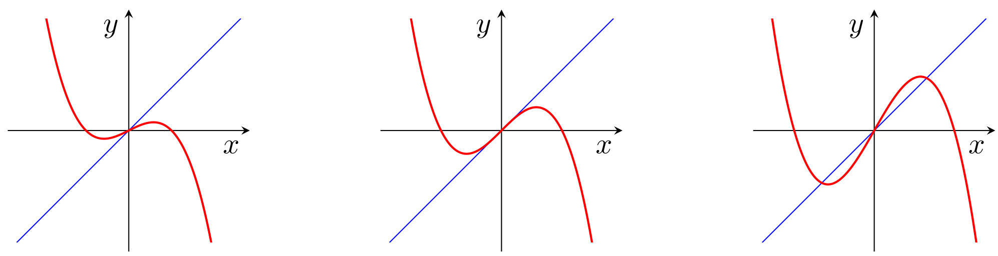
So in a neighborhood around $x = 0$, say $|x| < 1$, it follows that $\left|1 - x^2\right| < 1$. Therefore, for all x such that $|x| < 1$, $|f_0(x)| < |x|$. By Proposition 3.8, the fixed point at $x = 0$ is stable and attracting.

Case 3: $a > 0$. We have all three fixed points $\{0, \pm\sqrt{a}\}$. Thankfully, the stability of these points are easy to determine.

- $x = 0$. Here, $|f_a'(0)| = |1 + a| = 1 + a > 1$ for all $a > 1$. Hence the fixed point is repelling and therefore unstable.
- $x = \pm\sqrt{a}$. In this case, $|f_a'(\pm\sqrt{a})| = |1 - 2a|$, so that the fixed points are both stable and attracting for $0 < a < 1$, and (both) unstable and repelling for $a > 1$.

Like with the transcritical bifurcation, there is a fixed point for all values of a that changes stability. However, at the bifurcation point—or maybe *tri*furcation point is better—two new fixed points appear and the other fixed point, in this case $x = 0$, changes stability. The bifurcation diagram demonstrates this phenomenon and clarifies the name "pitchfork."

To see the symmetry in this example, consider the change of variables $x \mapsto -x$ in the continuous system $x' = f_a(x)$. Since the system is unchanged by this map (since the negative signs cancel), we say the system is *invariant* to this change of variable, and thus, it is invariant to this symmetry. □



(a) $a < 0$. (b) $a = 0$. (c) $a > 0$.

Figure 4.9: Plots of three graphs f_a for the three cases considered in Example 4.7.

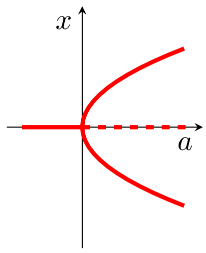


Figure 4.10: The pitchfork bifurcation occurring in Example 4.7.

Theorem 4.8 (Pitchfork bifurcation (special case))**.** *Let $F : J \times I \to \mathbb{R}$ be a C^3-function. Suppose that there is a point $(\bar{a}, \bar{x}) \in J \times I$ such that*

1. $F(a, \bar{x}) = \bar{x}$ *for all* $a \in J$*;*
2. $\frac{\partial F}{\partial x}(\bar{a}, \bar{x}) = 1$*;*
3. $\frac{\partial^2 F}{\partial x^2}(\bar{a}, \bar{x}) = 0$*;*
4. $\frac{\partial^2 F}{\partial x \partial a}(\bar{a}, \bar{x}) \neq 0$*; and*
5. $\frac{\partial^3 F}{\partial x^3}(\bar{a}, \bar{x}) \neq 0$.

Then there exists an open interval $U \subseteq I$ containing $\bar{x}$ and a unique function $h : U \to J$ such that $h(\bar{x}) = \bar{a}$, $F(h(x), x) = x$, $h'(\bar{x}) = 0$, $h''(\bar{x}) \neq 0$ and $h(x) \neq \bar{a}$ for all $x \in U \setminus \{\bar{x}\}$.

Proof outline. We proceed like we did in the proof of Theorem 4.6. Set

$$H(a, x) := \begin{cases} \frac{F(a,x)-x}{x-\bar{x}} & x \neq \bar{x}, \\ \lim_{x\to\bar{x}} \frac{F(a,x)-x}{x-\bar{x}} & x = \bar{x}. \end{cases}$$

We need to show that we can apply Implicit Function Theorem to H to construct the required function h such that $F(h(x), x) = x$ in a neighborhood U of $\bar{x}$. Calculate the derivatives of h via implicit differentiation. To see that $h'(x) = 0$, begin with the expression $H(h(x), x) = 0$ and differentiate with respect to x.

To see that $h''(\bar{x}) \neq 0$, we differentiate $F(h(x), x) = x$ with respect to x. (To simplify notation here, we write F_x for $\frac{\partial F}{\partial x}$ and so on.) We obtain

$$F_x(h(x), x) + F_a(h(x), x)h'(x) = 1.$$

Implicitly differentiating again with respect to x, we obtain

$$F_{xx} + F_{ax}h' + (F_{ax} + F_{aa}h')h' + F_a h'' = 0. \tag{4.6}$$

Evaluating at $\bar{x}$ (and using $h'(\bar{x}) = 0$) gives

$$F_a(\bar{a},\bar{x})h''(\bar{x}) = -F_{xx}(\bar{a},\bar{x}) = 0,$$

which doesn't help, as we have assumed that $F_a(\bar{a},\bar{x}) = 0$. Thus we differentiate (4.6) again with respect to x. After simplifying and substituting $x = \bar{x}$,

$$F_{xxx}(\bar{a},\bar{x}) + 3F_{ax}(\bar{a},\bar{x})h''(\bar{x}) = 0.$$

Thus,

$$h''(\bar{x}) = -\frac{1}{3}\frac{F_{xxx}(\bar{a},\bar{x})}{F_{ax}(\bar{a},\bar{x})} \neq 0 \tag{4.7}$$

as required. □

Note that there exist two intervals, say $J_1 = (a_1, \bar{a})$ and $J_2 = (\bar{a}, a_2)$, with $a_1 < \bar{a} < a_2$ such that in one of the intervals there exists exactly one fixed point and in the other there exist exactly three fixed points. We look at the stability property of the fixed points, and like in the transcritical bifurcation, there are two cases to consider: (1) the fixed point $x = \bar{x}$ for all a and (2) the fixed points x such that $h(x) = a$.

Case 1: $x = \bar{x}$. We look at the Taylor expansion of $\frac{\partial F}{\partial x}(a,\bar{x})$ about the point $a = \bar{a}$:

$$\begin{aligned}\frac{\partial F}{\partial x}(a,\bar{x}) &= \frac{\partial F}{\partial x}(\bar{a},\bar{x}) + \frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x})(a-\bar{a}) + O\left((a-\bar{a})^2\right)\\ &= 1 + \frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x})(a-\bar{a}) + O\left((a-\bar{a})^2\right).\end{aligned}$$

Because $\frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x}) \neq 0$, the stability changes at $a = \bar{a}$. This gives a precise means to analyze exactly how the stability changes.

- If $\frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x}) > 0$ then $\begin{cases} \bar{x} & \text{stable and attracting for } a < \bar{a}, \\ \bar{x} & \text{unstable and repelling for } a > \bar{a}. \end{cases}$
- If $\frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x}) < 0$ then $\begin{cases} \bar{x} & \text{stable and attracting for } a > \bar{a}, \\ \bar{x} & \text{unstable and repelling for } a < \bar{a}. \end{cases}$

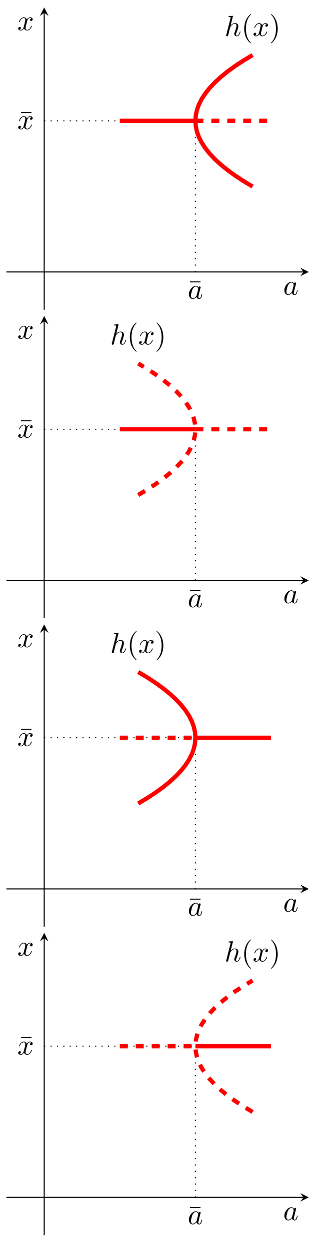
Case 2: $a = h(x)$. We will use the Taylor series expansion of $\frac{\partial F}{\partial x}(h(x),x)$ around $\bar{x}$ to see how the derivative is changing. Using $h'(\bar{x}) = 0$, $\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x}) = 0$, and (4.7), we get

$$\begin{aligned}\frac{\partial F}{\partial x}(h(x),x) &= \frac{\partial F}{\partial x}(\bar{a},\bar{x}) + \left(\frac{\partial^2 F}{\partial a\partial x}(\bar{a},\bar{x})h'(\bar{x}) + \frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})\right)(x-\bar{x})\\ &\quad + \frac{1}{2}\left[\frac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x}) + 2\frac{\partial^3 F}{\partial x^2\partial a}(\bar{a},\bar{x})h'(\bar{x}) + \frac{\partial^3 F}{\partial x\partial a^2}(\bar{a},\bar{x})\big(h'(\bar{x})\big)^2\right.\\ &\qquad\left. + \frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x})h''(\bar{x})\right](x-\bar{x})^2 + O\left((x-\bar{x})^3\right)\\ &= 1 + \frac{1}{2}\left[\frac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x}) + \frac{\partial^2 F}{\partial x\partial a}(\bar{a},\bar{x})h''(\bar{x})\right](x-\bar{x})^2 + O\left((x-\bar{x})^3\right)\\ &= 1 + \frac{1}{3}\frac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x})(x-\bar{x})^2 + O\left((x-\bar{x})^3\right).\end{aligned}$$

That is, near $x = \bar{x}$, we have the approximation:

$$\frac{\partial F}{\partial x}(h(x),x) \approx 1 + \frac{1}{3}\frac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x})(x-\bar{x})^2. \tag{4.8}$$

Equation (4.8) shows that both fixed points have the same stability property because the square terms $(x-\bar{x})^2$ is always nonnegative. Therefore, the stability of these fixed points are opposite to that of $\bar{x}$. The four possibilities of pitchfork bifurcations are displayed in Figure 4.11.



$$\left.\begin{array}{l} \dfrac{\partial^2 F}{\partial a \partial x}(\bar{a},\bar{x}) > 0 \\[2ex] \dfrac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x}) < 0 \end{array}\right\} \Rightarrow \quad h''(\bar{x}) > 0$$

$$\left.\begin{array}{l} \dfrac{\partial^2 F}{\partial a \partial x}(\bar{a},\bar{x}) > 0 \\[2ex] \dfrac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x}) > 0 \end{array}\right\} \Rightarrow \quad h''(\bar{x}) < 0$$

$$\left.\begin{array}{l} \dfrac{\partial^2 F}{\partial a \partial x}(\bar{a},\bar{x}) < 0 \\[2ex] \dfrac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x}) < 0 \end{array}\right\} \Rightarrow \quad h''(\bar{x}) < 0$$

$$\left.\begin{array}{l} \dfrac{\partial^2 F}{\partial a \partial x}(\bar{a},\bar{x}) < 0 \\[2ex] \dfrac{\partial^3 F}{\partial x^3}(\bar{a},\bar{x}) > 0 \end{array}\right\} \Rightarrow \quad h''(\bar{x}) > 0$$

Figure 4.11: Four possibilities of pitchfork bifurcations depending on the values of second and third partial derivatives of $F(a,x)$.

4.4 Period doubling bifurcation

The **period doubling bifurcation** does exactly as it sounds: the bifurcation point doubles the period of the system. The typical suspect for this kind of bifurcation is the logistic map.

Example 4.9. We return again to the logistic map $f_a(x) = ax(1-x)$ for $a \neq 0$. Recall that we have fixed points

- $x = 0$, which is stable for $a < 1$ and unstable for $a > 1$ and
- $x = \frac{a-1}{a}$, which is stable for $1 < a < 3$; unstable for $a < 1$ and $a > 3$.

This time, we examine what happens at $a = 3$. We know that the fixed point $x = \frac{2}{3}$ changes its stability property at $a = 3$, but perhaps something else also happens here. To investigate this, we begin by looking for periodic points of f_a:

$$f_a^2(x) = a(ax(1-x))(1-(ax(1-x))) = -a^3x^4 + 2a^3x^3 - a^3x^2 - a^2x^2 + a^2x.$$

The periodic points of f_a of period two are the fixed points of f_a^2. Simplifying and then factorizing $x = f_a^2(x)$, the fixed points must satisfy

$$0 = x\left(x - \frac{a-1}{a}\right)\left(x^2 - \left(\frac{a+1}{a}\right)x + \frac{a+1}{a^2}\right).$$

Since we know that $x = 0$ and $x = \frac{a-1}{a}$ are already fixed points of f_a, we focus the other factor. Thankfully, it is just a quadratic polynomial, so we can immediately write down the solutions. After some simplifications, the two additional fixed points are

$$x = \frac{a+1 \pm \sqrt{(a+1)(a-3)}}{2a}.$$

Since $(a+1)(a-3) < 0$ when $-1 < a < 3$, there are only the initial two fixed points of $f_a(x)$ in this interval. For $a = 3$, there is a unique solution, namely $x = 2/3$, and for $a > 3$, there are two solutions. By calculating the derivative of $f_a^2(x)$ with respect to x at these points, we can see that there is an $r > 0$ such that for $a \in (3, 3+r)$, they are stable fixed points of f^2. Hence they give rise to stable periodic orbits of f of minimal period 2. Thus, we extend the bifurcation diagram to include these, as seen in Figure 4.12.

With further analysis of the logistic map, one can show that additional period doubling bifurcation occurs for $a > 3$. The bifurcation diagram of the logistic map is given in Figure 4.13.
□

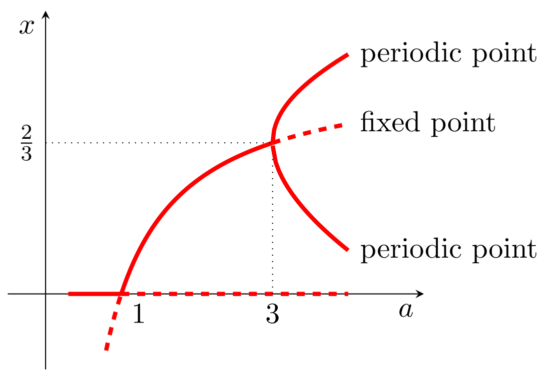


Figure 4.12: A snapshot of the bifurcation diagram from Example 4.9, given by the logistic map with $a = 3$.

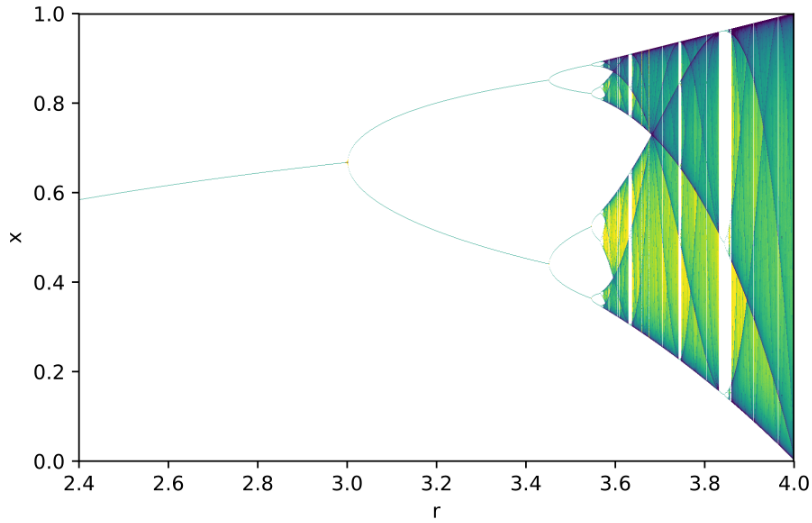


Figure 4.13: The bifurcation diagram of the logistic map $f_r(x) = rx(1-x)$.

In the next theorem we write $F^2(a,x) := F(a, F(a,x))$, which should be interpreted as $f_a^2(x)$.

Theorem 4.10 (Period doubling bifurcation)**.** *Let $F : J \times I \to \mathbb{R}$ be a C^3-function and define $F^2(a,x) := F(a,F(a,x))$ for $(a,x) \in J \times I$. Suppose that for $(\bar{a},\bar{x}) \in J \times I$, the following hold.*

1. $F(\bar{a},\bar{x}) = \bar{x}$;
2. $\frac{\partial F}{\partial x}(\bar{a},\bar{x}) = -1$;
3. $\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x}) = 0$;
4. $\frac{\partial F^2}{\partial a}(\bar{a},\bar{x}) = 0$;
5. $\frac{\partial^2 F^2}{\partial a \partial x}(\bar{a},\bar{x}) \neq 0$;
6. $\frac{\partial^3 F^2}{\partial x^3}(\bar{a},\bar{x}) \neq 0$.

Then there exists an interval $U \subseteq J$ and a unique C^3-function $g : U \to I$ such that $g(\bar{a}) = \bar{x}$, $F(a,g(a)) = g(a)$ for $a \in U$. In addition there exists an interval $V \subseteq I$ and a unique C^2-function $h : V \to I$ such that $h(\bar{x}) = \bar{a}$, $F^2(h(x),x) = x$, $h'(\bar{x}) = 0$ and $h''(\bar{x}) \neq 0$.

Idea. Apply the Implicit Function Theorem to F and Theorem 4.8 to the function F^2. □

Now we interpret Theorem 4.10. The function $g(a) = x$ describes the fixed point "prong" of the pitchfork, and the function $h(x) = a$ describes the periodic points of period two—that is, the other two "prongs" of the pitchfork. The fixed point $x = g(a)$ is a stable (resp. unstable) fixed point of f_a if and only if it is a stable (resp. unstable) fixed point of f_a^2. Note that

$$\frac{\partial F^2}{\partial x}(a,x) = \frac{\partial F}{\partial x}(a,F(a,x))\frac{\partial F}{\partial x}(a,x).$$

Therefore, since $F(a,g(a)) = x = g(a)$,

$$\frac{\partial F^2}{\partial x}(a,g(a)) = \left(\frac{\partial F}{\partial x}(a,g(a))\right)^2.$$

Thus,

$$\frac{\partial F^2}{\partial x}(\bar{a},\bar{x}) = 1, \qquad \frac{\partial^2 F^2}{\partial a \partial x}(\bar{a},\bar{x}) \neq 0,$$

so the stability changes at $(\bar{a},\bar{x})$. In our example, we see that in the region around $a = 3$, the behavior is like a pitchfork bifurcation. The fixed point $x = g(a)$ changes stability at $a = \bar{a}$. In a sufficiently small interval around $\bar{a}$, say $a < \bar{a} < b$, there are either two periodic points in $a < \bar{a}$ or in $\bar{a} < b$, and in the other interval there are no periodic points.

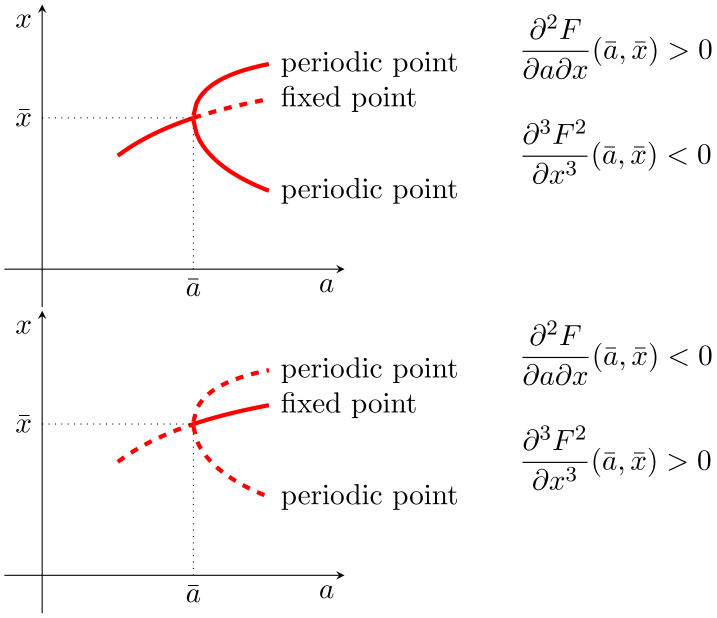


Figure 4.14: Locally around the point $(\bar{a}, \bar{x})$, the diagram look like a pitchfork bifurcation.

4.5 Miscellaneous types of bifurcations

In the previous sections of this chapter, we considered some of the most well-known and widely occurring types of bifurcations. In this section, we go through two short examples of additional phenomena that may occur.

Example 4.11 (*Generalized saddle-node bifurcation*)**.** Let $f_a(x) = x - (x^2 - a)(x^2 - 4a)$. The fixed points of $f_a(x)$ are $\{\pm\sqrt{a}, \pm 2\sqrt{a}\}$. For $a < 0$, there are no fixed points, and there is exactly 1 fixed point when $a = 0$. For $a > 0$, there are precisely 4 fixed points. Like with the saddle-node bifurcation in Section 4.1, a number of fixed points appear from (or disappear to) nothing.

We look at the qualitative behavior of these fixed points in a neighborhood around $a = 0$ by considering the derivative, $f_a'(x) = 1 + 10ax - 4x^3$.

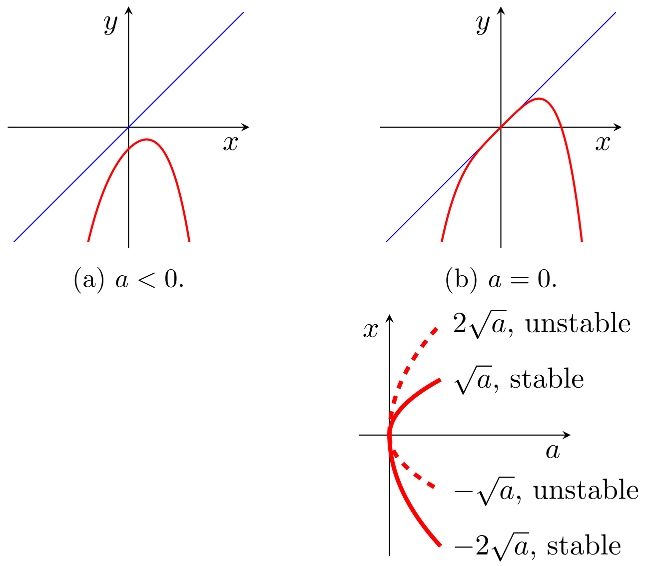
- $x = \sqrt{a}$. Then $|f_a'(\sqrt{a})| = |1 + 6a\sqrt{a}| > 1$ for all $a > 0$, so $x = \sqrt{a}$ is a repelling and unstable fixed point.
- $x = -\sqrt{a}$. Then $|f_a'(-\sqrt{a})| = |1 - 6a\sqrt{a}| < 1$ for positive a in a neighborhood of 0 (e.g. for $0 < a < 1/3$), so $x = -\sqrt{a}$ is an attracting and stable fixed point.
- $x = 2\sqrt{a}$. Then $|f_a'(2\sqrt{a})| = |1 - 12a\sqrt{a}| < 1$ for positive a in a neighborhood of zero, so $x = 2\sqrt{a}$ is an attracting and stable fixed point.
- $x = -2\sqrt{a}$. Then $|f_a'(-2\sqrt{a})| = |1 + 12a\sqrt{a}| > 1$ for all $a > 0$ so $x = -2\sqrt{a}$ is a repelling and unstable fixed point.

Note that for $a = 0$, the fixed point $x = 0$ is unstable. We plot $f_a(x)$ for different values of a in Figure 4.15, and the bifurcation diagram can be seen in Figure 4.15d. □

Example 4.12 (*Generalized pitchfork bifurcation*)**.** Let $f_a(x) = x - x(x^2 - a)(x^2 - 4a)$. The fixed points of $f_a(x)$ are $\{0, \pm\sqrt{a}, \pm 2\sqrt{a}\}$. For $a \leq 0$, we have only one fixed point, and for $a > 0$, we have 5 fixed points. As in Example 4.11, we look at the qualitative behavior of these fixed points in a neighborhood of 0 by considering the derivative, $f_a'(x) = 1 - 4a^2 + 15ax^2 - 5x^4$.

- $x = 0$. Then $|f_a'(0)| = |1 - 4a^2| < 1$ for all $a \neq 0$ in a neighborhood around $a = 0$, so in this region, $x = 0$ is an attracting and stable fixed point. Further analysis shows that for $a = 0$ the fixed point is also stable.

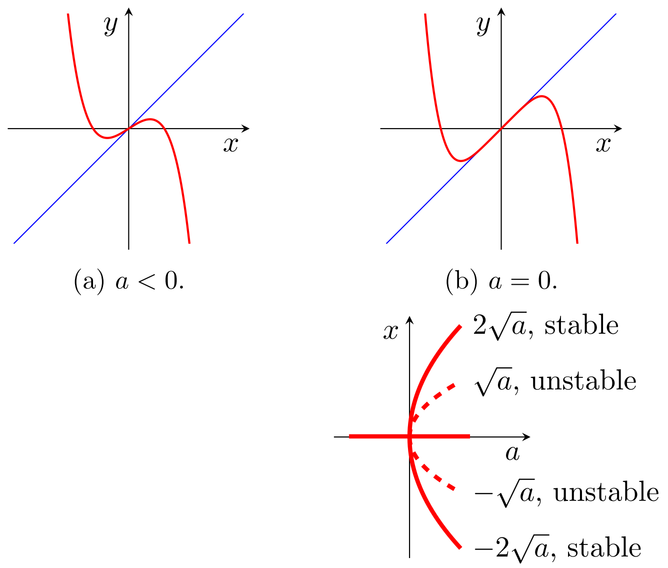
- $x = \pm\sqrt{a}$. Then $|f_a'(\pm\sqrt{a})| = |1 + 6a^2| > 1$ for all $a \neq 0$, so the fixed points $x = \pm\sqrt{a}$ are repelling and unstable.
- $x = \pm 2\sqrt{a}$. Then $|f_a'(\pm 2\sqrt{a})| = |1 - 24a^2| < 1$ for all $a \neq 0$ in a neighborhood around $a = 0$, so the fixed points $x = \pm 2\sqrt{a}$ are attracting and stable. $\square$



(a) $a < 0$. (b) $a = 0$. (c) $a > 0$.

(d) The bifurcation diagram.

Figure 4.15: Plots from Example 4.11 for different values of a together with its bifurcation diagram.



(a) $a < 0$. (b) $a = 0$. (c) $a > 0$.

(d) The bifurcation diagram.

Figure 4.16: We plot $f_a(x)$ from Example 4.12 for different values of a together with its bifurcation diagram.

Chapter 5

Linear discrete dynamical systems

Let A be a $d \times d$ matrix and define $f : \mathbb{R}^d \to \mathbb{R}^d$ by $f(x) = Ax$. This function gives rise to the linear dynamical system

$$x_{n+1} = Ax_n,$$

which we write as $x_n = A^n x_0$. As usual, the fixed points of the system satisfy $Ax = x$. Thus, we have either

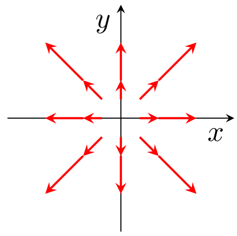
(i) $x = 0$ or

(ii) x is an eigenvector of A with eigenvalue 1.

Example 5.1. We look at the behavior of the system for different functions f (or matrices A). Throughout, we use a fixed basis (x, y) for $\mathbb{R}^2$.

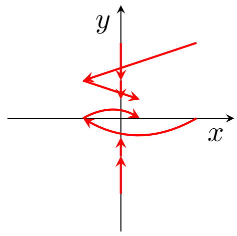
1. Expansion in two directions.

   $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,
   Eigenvectors: x, y,
   Eigenvalues: 2, 2.



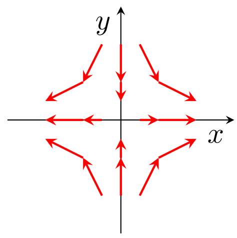
2. Contractions and change of sign

   $A = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$,
   Eigenvectors: x, y,
   Eigenvalues: $-1/2$, $1/2$.



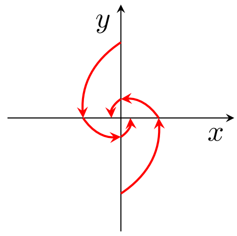
3. Expansion and contraction

   $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$,
   Eigenvectors: x, y,
   Eigenvalues: 2, $1/2$.



4. Rotation and contraction

$A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$,
Eigenvectors: $-\frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ i \end{pmatrix}$,
Eigenvalues: $-i/2$, $i/2$.



The rotation and contraction from this matrix can be more easily seen by rewriting it as

$$A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix}.$$

Note that the matrix on the right is standard form of a matrix that rotates x and y by $\theta = \pi/2$. $\square$

5.1 Matrix norms

We want to understand the behavior of dynamical systems of the form $x_n = A^n x_0$. In order to discuss stability, we need notions of distance, so our immediate goal is to transfer ideas of norms on vector spaces to matrices in a useful way.

Fix a basis $(e_1, \dots, e_d)$ of $\mathbb{R}^d$, and we assume throughout that $x \in \mathbb{R}^d$ is written as $x = \sum x_i e_i$, where $x_i \in \mathbb{R}$. Recall the standard inner product (or "dot product") of two vectors $x, y \in \mathbb{R}^d$ is

$$\langle x, y \rangle := \sum_{i=1}^{d} x_i y_i.$$

This gives rise to the L^2-norm $\|\cdot\|_2$ on $\mathbb{R}^d$ given by

$$\|x\|_2 := \langle x, x \rangle^{1/2} = \sqrt{\sum_{i=1}^{d} |x_i|^2}.$$

Note that this norm corresponds with the usual concept of Euclidean distance in $\mathbb{R}^d$. Therefore, we will drop the $_2$, and take $\|\cdot\|$ to always be the Euclidean norm. Treating matrices as vectors in a vector space, naturally transfers this norm to matrices in an entry-wise fashion.

Definition 5.2. The matrix norm $\|\cdot\|$ on $\mathrm{Mat}_{d\times d}(\mathbb{R})$ is given by

$$\|A\| = \sqrt{\sum_{i,j=1}^{d} |a_{ij}|^2}.$$

Proposition 5.3. *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, then for all $x \in \mathbb{R}^d$,*

$$\|Ax\| \le \|A\| \, \|x\|.$$

Proof. For $i \in \{1, \dots, d\}$, denote the ith row of A by A_i, and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^d$. Then Ax is the vector with ith entry equal to $\langle A_i, x \rangle$. Thus,

$$\begin{aligned} \|Ax\|^2 = \sum_{i=1}^{d} \langle A_i, x \rangle^2 &\le \sum_{i=1}^{d} \left(\langle A_i, A_i \rangle \langle x, x \rangle \right) \\ &= \langle x, x \rangle \sum_{i=1}^{d} \sum_{j=1}^{d} (A_i)_j (A_i)_j = \|A\|^2 \|x\|^2. \end{aligned}$$ $\square$

5.2 Jordan blocks

Eigenvalues and eigenvectors help us understand the action of a d-dimensional matrix A on a vector $x \in \mathbb{R}^d$. Furthermore, by using eigenvalues to represent a matrix in diagonal or block diagonal form, we can more easily study the repeated action of a matrix A on a vector x—that is, $A^n x$ for $n \in \mathbb{N}$.

Definition 5.4. A square matrix J is called a **Jordan block** if it is of the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda & 1 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix},$$

for $\lambda \in \mathbb{C}$. That is, J has λ along its diagonal and 1 along its super-diagonal. A square matrix A is said to be in **Jordan normal form** if it is of the form

$$A = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & J_m \end{pmatrix},$$

where each J_i is a Jordan block.

Theorem 5.5 (Jordan normal form)**.** *For any square matrix A there exists an invertible matrix S such that $S^{-1}AS$ is in Jordan normal form.*

Example 5.6. Consider the matrices

$$A = \begin{pmatrix} 2 & 4 & -6 & 0 \\ 4 & 6 & -3 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & -6 & 2 \end{pmatrix}, S = \begin{pmatrix} 1 & -1/4 & 0 & 1 \\ 0 & 1/4 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse of S is

$$S^{-1} = \begin{pmatrix} -1 & -1 & 3/2 & 2 \\ -4 & 0 & 0 & 4 \\ 0 & 0 & 1/2 & 0 \\ 1 & 1 & -3/2 & -1 \end{pmatrix},$$

and A is conjugated into Jordan normal form via S:

$$J = S^{-1}AS = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

The entries on the diagonal of the Jordan normal form of A are the eigenvalues of A. Thus, if A has a complex eigenvalue, the matrix $S^{-1}AS$ is complex. But for some purposes, it is useful to exclusively use real matrices.

Definition 5.7. A square matrix J is called a **real Jordan block** if it is either a Jordan block, as in Definition 5.4, with $\lambda \in \mathbb{R}$, or has the form

$$\begin{pmatrix} B & I & 0 & \dots & 0 \\ 0 & B & I & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & B & I \\ 0 & \dots & 0 & 0 & B \end{pmatrix},$$

where $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, $a, b \in \mathbb{R}$ and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

A square matrix A is in **real Jordan normal form** if it is of the form

$$A = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & J_m \end{pmatrix},$$

where each J_i is a real Jordan block.

Theorem 5.8. *For any real square matrix A there exists a real invertible matrix S such that $S^{-1}AS$ is in real Jordan normal form.*

The Jordan normal form allows us to easily calculate powers of a matrix because Jordan blocks are sparse matrices. The next lemma demonstrates this.

Lemma 5.9. *Let $J \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ be a Jordan block with eigenvalue λ. If $|\lambda| < 1$, then the sequence $(J^n)_{n=0}^{\infty}$ converges to the zero matrix. If $|\lambda| > 1$, then the sequence diverges in $\mathrm{Mat}_{d\times d}(\mathbb{C})$.*

Proof. By induction it follows that the nth powers of J are

$$J^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \dots & \binom{n}{d-1}\lambda^{n-d+1} \\ & \lambda^n & \binom{n}{1}\lambda^{n-1} & \dots & \binom{n}{d-2}\lambda^{n-d+2} \\ & & \ddots & \ddots & \vdots \\ & & & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & & & & \lambda^n \end{pmatrix}.$$

For every $k \in \{0, \dots, d-1\}$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leq \frac{n^k}{k!}.$$

Fix a pair (i, j) such that $i \leq j$. Set $k = j - i$, so the (i, j)-entry of J^n has modulus at least as large as $|\lambda|^{n-k}$ and at most

$$\left|\binom{n}{k}\lambda^{n-k}\right| = \binom{n}{k}|\lambda|^{n-k} \leq \frac{n^k\,|\lambda|^{n-k}}{k!}.$$

If $|\lambda| < 1$, then for fixed k,

$$\lim_{n\to\infty} \frac{n^k\,|\lambda|^{n-k}}{k!} = 0. \tag{5.1}$$

It follows from (5.1) that when $|\lambda| < 1$, the sequence $(J^n)_{n=0}^{\infty}$ converges to the zero matrix. On the other hand, if $|\lambda| > 1$, then $|\lambda|^{n-k} \to \infty$ as $n \to \infty$. Therefore, when $|\lambda| > 1$, the modulus grows without bound, and hence the sequence does not converge since the (i, j)-entry does not converge in $\mathbb{C}$. □

The case that $|\lambda| = 1$ is less direct. Observe that the powers of the Jordan block $J = (1)$ trivially converges to J, but the powers of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ does not converge since the $(1, 2)$-entry grows without bound.

Theorem 5.10. *If A be a square matrix where every eigenvalue λ satisfies $|\lambda| < 1$, then*

$$\lim_{n\to\infty} A^n = 0.$$

Proof. We begin by applying Theorem 5.5 to get A into Jordan normal form, $A = SJS^{-1}$. Thus

$$A^n = \left(SJS^{-1}\right)\left(SJS^{-1}\right)\cdots\left(SJS^{-1}\right) = SJ^nS^{-1}. \tag{5.2}$$

Then (5.2) implies that

$$\lim_{n\to\infty} A^n = 0 \Longleftrightarrow \lim_{n\to\infty} J^n = 0.$$

Hence, we just consider the asymptotic behavior of J. If

$$J = \begin{pmatrix} J_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_m \end{pmatrix},$$

then for $n \in \mathbb{N}$,

$$J^n = \begin{pmatrix} J_1^n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_m^n \end{pmatrix}.$$

Apply Lemma 5.9 to each Jordan block. Since every eigenvalue of A satisfies $|\lambda| < 1$, it follows that each block converges to the zero matrix. Hence, J^n converges to the zero matrix. □

5.3 Stability criteria

Now we use the theory from the previous sections to answer stability questions concerning linear dynamical systems.

Proposition 5.11. *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ such that all eigenvalues λ satisfy $|\lambda| < 1$, then $x = 0$ is a stable and attracting fixed point of the dynamical system in $\mathbb{R}^d$ defined by $x_{n+1} = Ax_n$.*

Proof. That $x = 0$ is an attracting fixed point is a direct consequence of Theorem 5.10: for all $x_0 \in \mathbb{R}^d$,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} A^n x_0 = 0.$$

Thus, every orbit of x_0 converges to 0.

To see that the fixed point is also stable, let $\varepsilon > 0$. By Theorem 5.10, $\|A^n\| \to 0$. Thus, the sequence $(\|A^n\|)$ is bounded and

$$\alpha = \max_{n\in\mathbb{N}} \{1, \|A^n\|\}$$

is finite. For $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, Proposition 5.3 implies that $\|A^n x\| \le \|A^n\| \|x\|$. Then for $n \in \mathbb{N}$,

$$\|x_n\| = \left\|A^{n-1}x_0\right\| \le \left\|A^{n-1}\right\| \|x_0\| \le \alpha \|x_0\|,$$

so that for all $n \in \mathbb{N}$, $\|x_0\| < \varepsilon/\alpha =: \delta$ implies that $\|x_n\| < \varepsilon$. □

Theorem 5.12. *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ has an eigenvalue λ such that $|\lambda| > 1$, then there exists a vector $x \in \mathbb{R}^d$ such that*

$$\lim_{n\to\infty} \|A^n x\| = \infty.$$

Proof. If $\lambda \in \mathbb{R}$, then there is a real corresponding eigenvector x such that

$$\|A^n x\| = \|\lambda^n x\| = |\lambda|^n \|x\|.$$

Since $|\lambda| > 1$, as $n \to \infty$, $\|A^n x\| \to \infty$.

If $\lambda \in \mathbb{C}$, then write $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. Then let $w = x + iy$ be an eigenvector for λ, where $x, y \in \mathbb{R}^d$ are linearly independent. For every $n \in \mathbb{N}$, $A^n w = \lambda^n w$, so λ^n is an eigenvalue for A^n. Writing $\lambda = re^{i\theta}$, it follows that

$$\lambda^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Thus,

$$\begin{aligned} A^n x + iA^n y &= A^n w \\ &= \lambda^n w \\ &= r^n(\cos(n\theta) + i\sin(n\theta))(x + iy) \\ &= r^n(x\cos(n\theta) - y\sin(n\theta)) + ir^n(x\sin(n\theta) + y\cos(n\theta)). \end{aligned}$$

Considering the real and imaginary parts separately,

$$\begin{aligned} A^n x &= r^n(x\cos(n\theta) - y\sin(n\theta)), \\ A^n y &= r^n(x\sin(n\theta) + y\cos(n\theta)). \end{aligned}$$

Putting everything together and setting $\varphi = n\theta$,

$$\begin{aligned} \|A^n x\|^2 &= \langle r^n(x\cos\varphi - y\sin\varphi), r^n(x\cos\varphi - y\sin\varphi)\rangle \\ &= r^{2n}\left((\cos\varphi)^2\langle x, x\rangle + (\sin\varphi)^2\langle y, y\rangle - 2(\cos\varphi)(\sin\varphi)\langle x, y\rangle\right) \\ &\geq r^{2n}\left((\cos\varphi)^2\|x\|^2 + (\sin\varphi)^2\|y\|^2 - 2|\cos\varphi||\sin\varphi||\langle x, y\rangle|\right) \\ &= r^{2n}\left((|\cos\varphi|\|x\| - |\sin\varphi|\|y\|)^2 + 2|\cos\varphi||\sin\varphi|(\|x\|\|y\| - |\langle x, y\rangle|)\right). \end{aligned}$$

By the Cauchy–Schwarz inequality, $\|x\|\|y\| - |\langle x, y\rangle| > 0$. From above and since $r = |\lambda| > 1$, it follows that

$$\lim_{n\to\infty} \|A^n x\| = \infty.$$

□

This leads directly to the following corollaries.

Corollary 5.13. *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ has an eigenvalue λ with $|\lambda| > 1$, then $x = 0$ is an unstable fixed point of the system $x_{n+1} = Ax_n$.*

The next corollary is not as immediate, but we do not prove it here.

Corollary 5.14. *Let $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ such that every eigenvalue λ satisfies $|\lambda| > 1$. Then for all nonzero $x \in \mathbb{R}^d$,*

$$\lim_{n\to\infty} \|A^n x\| = \infty,$$

and $x = 0$ is a repelling fixed point of the system $x_{n+1} = Ax_n$.

Remark 5.15. If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ such that every eigenvalue λ satisfies $|\lambda| > 1$, then 0 is not an eigenvalue of A, so A is invertible. Since the eigenvalues of A^{-1} are $1/\lambda$, where λ is an eigenvalue of A, it follows that that $\lim_{n\to\infty} A^{-n}x = 0$.

Theorem 5.16. *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ with no eigenvalues λ such that $|\lambda| \in \{0, 1\}$, then there exist subspaces V_s and V_u of $\mathbb{R}^d$, with $\dim V_s + \dim V_u = d$, satisfying the following.*

1. *If $x \in V_s$, then $\lim_{n\to\infty} A^n x = 0$.*

2. *If $x \in V_u$, then $\lim_{n\to\infty} A^{-n} x = 0$.*

3. If $x \notin V_s$, then $\lim_{n\to\infty} \|A^n x\| = \infty$.

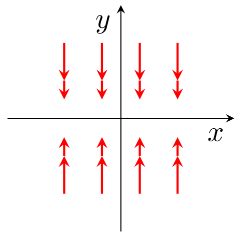
The subspace V_s from Theorem 5.16 is called the **stable subspace** and V_u the **unstable subspace** of the system defined by such a matrix A.

Example 5.17. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 5/2 \end{pmatrix}$, and define $x_{n+1} = Ax_n$. The eigenvalues of A are $1/2$ and 2; the corresponding eigenvectors are $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively. They form a basis for $\mathbb{R}^2$. Moreover, $V_s = \langle u \rangle$ and $V_u = \langle v \rangle$.

Example 5.18. Now we look at a few linear dynamical systems that have matrices with $\lambda = 1$ as an eigenvalue. We fix a basis (x, y) for $\mathbb{R}^2$.

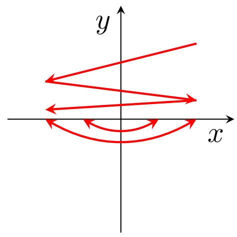
1. In this case there exists an eigenvector x such that $Ax = x$, so there exists a subspace of fixed points.

   $A = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$
   Eigenvectors: x, y
   Eigenvalues: 1, 1/2



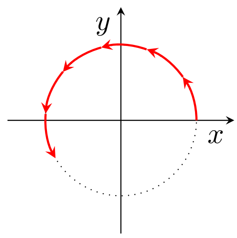
2. There exists an eigenvector x such that $Ax = -x$, that is $A^2 x = x$. Therefore there exists a subspace of periodic points of period 2.

   $A = \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}$,
   Eigenvectors: x, y,
   Eigenvalues: -1, $1/2$.



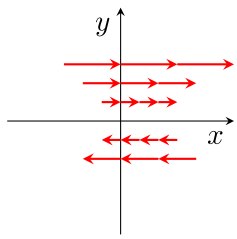
3. If $\varphi = q \cdot 2\pi$ with $q \in \mathbb{Q}$ then there exists a subspace of periodic points. If q is irrational, the orbits are dense on the circle.

   $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$,
   Eigenvectors: $-x + iy$, $x + iy$,
   Eigenvalues: $e^{i\varphi}$, $e^{-i\varphi}$.



4. Additional example for $\lambda = 1$. Here we also have a fixed subspace.

   $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,
   Eigenvectors: x,
   Eigenvalues: 1,



Chapter 6

Non-linear discrete dynamical systems

In this chapter we briefly consider dynamical systems defined by non-linear functions $f : X \to X$, where $X \in \mathbb{R}^d$, which we may write as

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_d(x) \end{pmatrix},$$

for $x = (x_1, \dots, x_d)^\top$. We may write functions $f : X \to X$ using row vectors, but these should be interpreted as column vectors. We will use $^\top$ to mean transpose.

As in the one-dimensional non-linear case (see Theorem 3.5), we can use derivatives of the function at a fixed point to analyse the behavior of the system around this point. Note that unadorned norms signify the L^2-norm, so $\|\cdot\| = \|\cdot\|_2$, see Section 5.1.

The *Jacobian* of f at $x \in X$ is the $(d \times d)$-matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_d}(x) \\ \vdots & & \vdots \\ \frac{\partial f_d}{\partial x_1}(x) & \dots & \frac{\partial f_d}{\partial x_d}(x) \end{pmatrix}.$$

If $\bar{x} \in X$ is a fixed point of f, then we have $f(\bar{x}) = \bar{x}$, and expanding f in a Taylor series around $\bar{x}$ gives

$$f(x) = \bar{x} + Df(\bar{x}) \cdot (x - \bar{x}) + O\left(\|x - \bar{x}\|^2\right).$$

Therefore, for x very close to $\bar{x}$,

$$f(x) \approx \bar{x} + Df(\bar{x})(x - \bar{x}).$$

Definition 6.1. Let $f : X \to X$, where $X \subseteq \mathbb{R}^d$, be a C^1-function. A fixed point $\bar{x}$ of f is called **hyperbolic** if for all eigenvalues λ of $Df(\bar{x})$, $|\lambda| \neq 1$.

The following theorem is the analog of Theorem 3.5 for d-dimenisonal non-linear systems.

Theorem 6.2. *Let f be a C^1-function and $\bar{x} \in X$ a fixed point of f. Then*

(i) The fixed point $\bar{x}$ is stable and attracting if $|\lambda| < 1$ for all eigenvalues λ of $Df(\bar{x})$.

(ii) The fixed point $\bar{x}$ is unstable if there exists an eigenvalue λ of $Df(\bar{x})$ such that $|\lambda| > 1$.

(iii) The fixed point $\bar{x}$ is unstable and repelling if $|\lambda| > 1$ for all eigenvalues λ of $Df(\bar{x})$.

Definition 6.3. Let f be a C^1-function and $\bar{x} \in X$ a fixed point of f. The fixed point $\bar{x}$ is called a **saddle point** if there exist eigenvalues λ and μ of $Df(\bar{x})$ such that $|\lambda| < 1 < |\mu|$.

Definition 6.4. A **local stable manifold** is a region ω^s such that

$$\lim_{n\to\infty} f^n(x) = \bar{x}$$

if $x \in \omega^s$ and a **local unstable manifold** is a region ω^u such that

$$\lim_{n\to\infty} f^{-n}(x) = \bar{x}$$

if $x \in \omega^u$. *Note that f^{-n} need not be defined globally; local existence is sufficient.*

In two dimensions, these manifolds can be seen as curves that go "in to" of "out of" the fixed point. Compare this with the definition of stable/ unstable sets from Section 2.3. If $\bar{x}$ is a saddle point, then there exist local stable and unstable manifolds.

The idea used to visualize these manifolds is to change variables; that is, for subsets $U, V \subseteq \mathbb{R}^d$, find $h : U \to V$, with $h(0) = \bar{x}$ such that $L = h^{-1} \circ f \circ h$ is a linear map in a (small) neighborhood of 0.

Example 6.5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = \left(\frac{1}{2}x,\ 2y - \frac{15}{8}x^3\right)^\top,$$

with Jacobian

$$Df(x, y) = \begin{pmatrix} 1/2 & 0 \\ -45x^2/8 & 2 \end{pmatrix}.$$

Since $f(0) = 0$, we see that f has the fixed point $\bar{x} = 0$. Moreover, as

$$Df(0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix},$$

it follows that $\bar{x}$ is a saddle point. Since $f(0, t) = (0, 2t)^\top$, then

$$f^n(0, t) = (0, 2^n t)^\top, \qquad f^{-n}(0, t) = \left(0, 2^{-n} t\right)^\top.$$

Thus the y-axis is the unstable manifold. Since

$$f(t, t^3) = \left(\frac{t}{2}, \frac{t^3}{8}\right)^\top = \left(\frac{t}{2}, \left(\frac{t}{2}\right)^3\right)^\top,$$

we see that the curve $y = x^3$ is the stable manifold. See Figure 6.1 for plots of these manifolds.

Now, we define $h(x, y) = (x, x^3 - y)^\top$, so $h^2(x, y) = (x, y)^\top$. Therefore, $h^{-1}(x, y) = h(x, y)$. Define $L := h^{-1} \circ f \circ h$. Then

$$L(x, y) = (h^{-1} \circ f \circ h) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/2 \\ 2y \end{pmatrix} = Df(0) \begin{pmatrix} x \\ y \end{pmatrix}. \quad \square$$

Remark 6.6. The behaviour of the orbit of a point x in the unstable manifold ω^u of a saddle point $\bar{x}$ is not fixed. Some typical situations are:

(i) the orbit of x converges to the saddle point $\bar{x}$—in this case we say that x is **homoclinic** or that it has a homoclinic orbit;

(ii) the orbit of x converges to a different saddle point $\bar{y}$—in this case we say that x is **heteroclinic** or that it has a heteroclinic orbit;

(iii) the orbit of x converges to a different attractor (not necessarily a fixed point), see Figure 6.2 for an example.

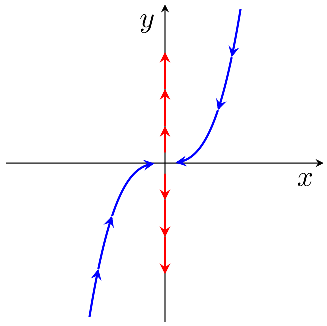


Figure 6.1: A few paths from Example 6.5. The red is a unstable manifold, and the blue is the stable manifold.

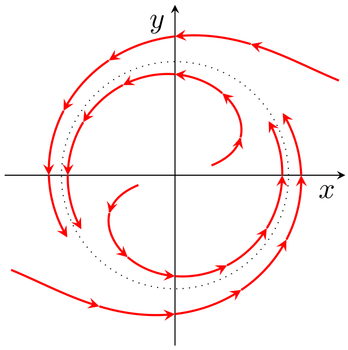


Figure 6.2: Orbits that are neither homoclinic nor heteroclinic.

6.1 An introduction to Lyapunov exponents

An important concept in the study of stability is that of Lyapunov exponents.[1] Informally, these numbers measure how two close, but different, paths diverge, and they can be used to describe whether or not a system is chaotic. Before we define the Lyapunov number and exponent, we bring in ideas from linear algebra.

Definition 6.7. Let A be a $d \times d$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_d$. The **spectral radius** of A is the number $\rho(A) = \max_i |\lambda_i|$.

We can rephrase earlier results in terms of the spectral radius for cleaner statements. Let $f : X \to X$ be continuously differentiable.

1. A fixed point $\bar{x} \in X$ of f is stable if $\rho(Df(\bar{x})) < 1$ and unstable if $\rho(Df(\bar{x})) > 1$.

2. A periodic point $\bar{x} \in X$ of period p is stable if $\rho(Df^p(\bar{x})) < 1$ and unstable if $\rho(Df^p(\bar{x})) > 1$. Thus, $\bar{x}$ is stable if $(\rho(Df^p(\bar{x})))^{1/p} < 1$ and unstable if $(\rho(Df^p(\bar{x})))^{1/p} > 1$.

We consider the spectral radius of the n-fold iteration of f. For $x_0 \in X$, by the chain rule,

$$Df^n(x_0) = Df(x_{n-1})Df(x_{n-2}) \cdots Df(x_1)Df(x_0). \tag{6.1}$$

If $x_0 = \bar{x}$ is a fixed point of f, then Equation (6.1) becomes $Df^n(\bar{x}) = (Df(\bar{x}))^n$, implying

$$\rho\left(Df(\bar{x})\right) = \left(\rho\left(Df^n(\bar{x})\right)\right)^{1/n}. \tag{6.2}$$

The quantity in Equation (6.2) is the main character in the definition of the Lyapunov number.

[1] In fact, Lyapunov *developed* the notion of stability in his PhD thesis in the late 1800s.

Definition 6.8. Let $X \subseteq \mathbb{R}^d$ and $f : X \to X$ define a dynamical system via $x_n = f(x_{n-1})$. Let $x_0 \in X$ such that $\rho(Df(x_n)) > 0$ for all $n \in \mathbb{N}$. We define the **Lyapunov number** of the orbit $O^+(x_0)$ to be, provided the limit exists,

$$N(x_0) = \lim_{n\to\infty} \left(\rho\left(Df^n(x_0)\right)\right)^{1/n}$$

and the **Lyapunov exponent** of $O^+(x_0)$ to be

$$\Lambda(x_0) = \log N(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \rho\left(Df^n(x_0)\right).$$

So the discussion prior to the definition of the Lyapunov number shows that if $x_0 = \bar{x}$ is a fixed point of f, then (6.2) implies

$$\Lambda(\bar{x}) = \ln \rho\left(Df(\bar{x})\right).$$

Similarly, if $x_0 = \bar{x}$ is a periodic point of f of period p, then

$$\Lambda(\bar{x}) = \frac{1}{p} \log \rho\left(Df^p(\bar{x})\right).$$

And in one dimension, when $X \subseteq \mathbb{R}$, then $\rho\big(Df^n(x_0)\big) = |f'(x_0)||f'(x_1)|\cdots|f'(x_{n-1})|$. We can rewrite this as

$$\Lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)|.$$

The Lyapunov exponent is used to estimate stability in the following way. Let x_0 and y_0 be two points close to each other. Using a Taylor expansion to approximate,

$$x_n - y_n = f^n(x_0) - f^n(y_0) \approx Df^n(x_0)(x_0 - y_0).$$

Now we use the Lyapunov exponent $\Lambda(x_0)$ to estimate $Df^n(x_0)$:

$$\|x_n - y_n\| \approx e^{\Lambda(x_0)n} \|x_0 - y_0\|,$$

so we can split the analysis into two cases based on the sign of $\Lambda(x_0)$.

- If $\Lambda(x_0) > 0$, then $e^{\Lambda(x_0)n}$ grows and the orbits separate (indicating instability).
- If $\Lambda(x_0) < 0$, then $e^{\Lambda(x_0)n}$ decays and the orbits get closer (indicating stability).

As we have seen in our examples, nearby orbits diverge from each other when being repelled, and nearby orbits get closer when being attracted. Importantly, separation can be an indicator of chaotic behavior, and so when $\Lambda(x_0) > 0$ one may consider the system to be chaotic.

Example 6.9. Consider the logistic map with $a = 4$, so $f(x) = 4x(1 - x)$. Let $x_0 = 1/4$. Then

$$x_1 = f(x_0) = f(1/4) = 3/4,$$
$$x_2 = f(x_1) = f(3/4) = 3/4.$$

Thus, the orbit is eventually fixed $O^+(1/4) = (1/4,\ 3/4,\ 3/4, \dots)$.

Now we consider the Lyapunov number at $x = 1/4$. Since $f'(x) = 4 - 8x$,

$$|f'(1/4)| = 2 = |f'(3/4)|.$$

Thus, $|f'(x_n)| = 2$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\Lambda(x_0) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log|f'(x_i)| \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(2) \\ &= \log(2) > 0.\end{aligned}$$

Because $\Lambda(1/4) > 0$, orbits starting close to $x = 1/4$ will diverge from one another. (Consider trying this for yourself.) We know from Theorem 3.5, the fixed point $x = 3/4$ is indeed unstable, so this description of the Lyapunov number matches with our understanding so far.

We can also calculate the Lyapunov exponent of the fixed point $x = 0$, which is $\Lambda(0) = \log(4) > 0$. Again, this indicates instability, and again from Theorem 3.5, we know that $x = 0$ is unstable. Notice that $0 < \Lambda(1/4) < \Lambda(0)$; might be be some way to interpret this? $\square$

For fixed points, we do not need to calculate the Lyapunov exponent as this will simply repeat the information that we have from the derivative. That is, if x is a fixed point such that $|f'(x)| < 1$, then the fixed point is stable, and correspondingly, $\Lambda(x) = \log|f'(x)| < 0$. If x is a fixed point such that $|f'(x)| > 1$, then we know that the fixed point is unstable, and correspondingly, $\Lambda(x) = \ln|f'(x)| > 0$. However, in higher-dimensional systems, this becomes much more useful.

6.2 Some remarks on chaotic behavior

Definition 6.10. Let $f : X \to X$ be a function. The orbit $O^+(x)$ of a point $x \in X$ is **stable** if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x - y\| < \delta$ implies that for all $n \in \mathbb{N}$,

$$\|f^n(x) - f^n(y)\| < \varepsilon.$$

The orbit is called **unstable** if it is not stable.

Note that the x in the above definition need not be a periodic point, however if it is, this definition coincides with Definition 2.27. If there exists $Y \subset X$ such that $f(Y) \subseteq Y$, then the orbit of $x \in Y$ can be stable with respect to Y assuming the orbit satisfies the above definition when $y \in Y$ and $\|x - y\| < \delta$. Similarly, the orbit may be unstable with respect to Y. Note that a stable orbit is stable with respect to all $Y \subseteq X$, and an orbit that is unstable with respect to some $Y \subseteq X$ is unstable.

Example 6.11. Consider the logistic map with parameter $a = 1$: that is, $f(x) = x(1-x)$. The orbit $O^+(0)$ is unstable as for all $y < 0$,

$$\lim_{n\to\infty} f^n(y) = -\infty.$$

However, the orbit of $y \in [0, 1]$ is stable with respect to $Y = [0, 1]$. To see this, let $\varepsilon > 0$, and choose $\delta = \varepsilon$. Then for $y \in Y$ with $y < \delta$,

$$|f(0) - f(y)| = |f(y)| = |y||1 - y| \leq |y| < \varepsilon. \quad \square$$

Definition 6.12. Let $f : X \to X$. We say that f has **sensitive dependence** if there exists an $\varepsilon > 0$ such that for any $x \in X$ and any $\delta > 0$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $\|x - y\| < \delta$ and

$$\|f^n(x) - f^n(y)\| \geq \varepsilon.$$

We could refine the above definition to consider $Y \subseteq X$. Assuming $f(Y) \subseteq Y$, then f has sensitive dependence on initial conditions *relative to* Y if every orbit starting in Y is unstable with respect to Y. Informally, this is the same as points that start close together in Y have large differences in orbit behavior.

Example 6.13. The logistic map with parameter $a = 4$, $f(x) = 4x(1 - x)$, has sensitive dependence on initial conditions in $[0, 1]$. See Figure 6.3 for an example.

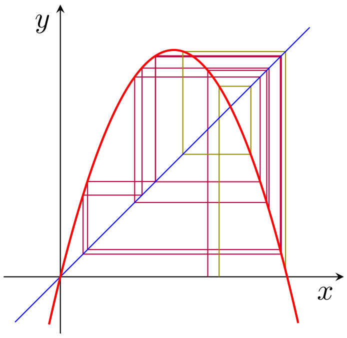


Figure 6.3: A demonstration of the chaotic nature of the logistic map for $a = 4$. Two paths are mapped: the purple starts at $x_0 = 0.65$ and the green starts at $x_0 = 0.7$.

Definition 6.14. Let $f : X \to X$. We say that f is **topologically mixing** if for every pair of non-empty open sets $U, V \subseteq X$ are non-empty, there exists $N \in \mathbb{N}$ such that for all $n > N$, $f^n(U)$ has non-empty intersection with $f^n(V)$.

A consequence of topological mixing is that *every* pair of non-empty open sets, no matter how small or fall apart, eventually always overlap by iteration of f. In other words, every non-empty open set has an orbit that intersects every open set of X.

Example 6.15. The logistic map with parameter $a = 4$, $f(x) = 4x(1 - x)$, is topologically mixing on $[0, 1]$. See Figure 6.4 for an example.

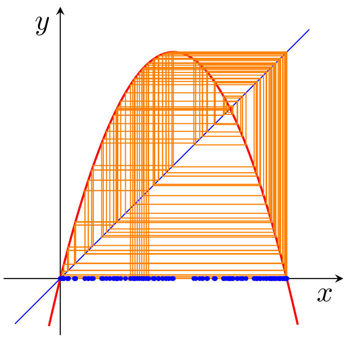


Figure 6.4: A demonstration of the topological mixing property of the logistic map for $a = 4$. Here, the open set is $(0.3, 0.4)$, and we plot 20 iterations of eight paths. We color points in the orbit on the x-axis in blue.

Recall that a set $Y \subseteq X$ is *dense* in X if for every $\varepsilon > 0$ and every $x \in X$, every ball $B_\varepsilon(x)$ contains elements of Y.

Example 6.16. Consider the logistic map with parameter $a = 4$, $f(x) = 4x(1 - x)$. Recall from the example in Section 1.1.3, that for $\varphi_0 \in \mathbb{R}$ and $x_0 = (\sin(\varphi_0))^2$, we have $x_n = f^n(x_0) = (\sin(2^n\varphi_0))^2$. If $s \in \mathbb{R} \setminus \mathbb{Q}$, then the orbit of $x_0 := (\sin(2\pi s))^2$ is dense in $[0, 1]$. Further, the set of all points $x \in [0, 1]$ whose orbits under f are periodic is dense in $[0, 1]$.

Definition 6.17. Let $f : X \to X$. We say that f is **chaotic** if

1. f is topologically mixing and
2. the periodic points of f are dense in X.

Since the logistic map f with $a = 4$ is both topologically mixing on $[0, 1]$ and has dense periodic points in $[0, 1]$, it follows that f is chaotic on $[0, 1]$.

The two conditions of Definition 6.17 imply that the map f is sensitive to initial conditions—a quality that we intuitively associate with chaotic behavior. However, a map that is sensitive to initial conditions need not be chaotic. If we do not have both conditions of Definition 6.17 satisfied, then chaotic behavior need not be present.

Chapter 7

Ordinary differential equations and flows

We now turn to the study of continuous dynamical systems, and thus to differential equations. In this chapter, we cover some basic aspects of ordinary differential equations (ODEs).

In its simplest form, an ODE is an equation involving the derivative of a single-variable function, the function itself, and the variable on which it depends. If x is a differentiable function of t and $t \in \mathbb{R}$, then a **first order ODE** has the form

$$x'(t) = f(t, x). \tag{7.1}$$

Such an equation defines a continuous dynamical system, where $x(t)$ is the state of the system at time t. If the function f in (7.1) is independent of the time t, then the system is called **autonomous** and can be written as $x' = g(x)$. Otherwise, the system is called **non-autonomous**.

By allowing higher order derivatives, we form higher order ODEs; an **ODE of order** n has the form

$$x^{(n)} = f\left(t, x, x', x'', \ldots, x^{(n-1)}\right). \tag{7.2}$$

A **system of ODEs** consists of two or more inter-connected ODEs. For example, if $\mathbf{x} = (x_1, \ldots, x_d)$, where each x_i depends only on t, then

$$\begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_d^{(n)} \end{pmatrix} = \begin{pmatrix} f_1\left(t, \mathbf{x}, \mathbf{x}', \ldots, \mathbf{x}^{(n-1)}\right) \\ f_2\left(t, \mathbf{x}, \mathbf{x}', \ldots, \mathbf{x}^{(n-1)}\right) \\ \vdots \\ f_d\left(t, \mathbf{x}, \mathbf{x}', \ldots, \mathbf{x}^{(n-1)}\right) \end{pmatrix}. \tag{7.3}$$

With vector-valued functions, we can rewrite the system in (7.3) as

$$\mathbf{x}^{(n)} = \mathbf{F}\left(t, \mathbf{x}, \mathbf{x}', \ldots, \mathbf{x}^{(n-1)}\right).$$

A higher order ODE may be transformed into a system of first order ODEs as follows. Given an ODE of order n as in (7.2), we set $x_1 := x$, $x_2 := x'$, …, and $x_n := x^{(n-1)}$. Then Equation (7.2) becomes

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1, x_2, \ldots, x_n) \end{pmatrix}.$$

Thus, it suffices to study systems of first order ODEs. Note that in general, a system of first order ODEs may also be written as

$$x(t) = f(t, x),$$

where $x : \mathbb{R} \to \mathbb{R}^d$ and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ (we denoted this above with bold).

Example 7.1. Consider the second order ODE:

$$x'' = \frac{3}{t}x' - \frac{4}{t^2}x + t.$$

By introducing the variables $x_1 = x$ and $x_2 = x'$, we transform the equation into the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= t - \frac{4}{t^2}x_1 + \frac{3}{t}x_2. \quad \square \end{aligned}$$

Given a first order system, we may be able to solve it, or at least determine whether or not a solution exists. We are interested in whether or not the solution $x(t)$ for a given initial condition $x(t_0) = x_0$ is unique, and how the solution changes as we vary the initial condition (that is, how the orbit varies as we vary the initial state).

We recall a couple of elementary methods of solving differential equations.

Example 7.2. (*Separation of variables*) Consider the differential equation given by

$$x' = -x^2 e^t. \tag{7.4}$$

Since the ODE in (7.4) can be written in the form $x' = p(x)q(t)$, it is *separable*. When $x \neq 0$, Equation (7.4) can be rewritten as

$$-\frac{1}{x^2}x' = e^t.$$

Integrating both sides with respect to t and solving for x yields

$$x = \frac{1}{e^t + C}.\square$$

We complete the analysis by also considering the case $x = 0$; this gives the solution $x = 0$. $\square$

Example 7.3. (*Integrating factor*) Consider the initial value problem (IVP)

$$\begin{cases} x' = -\frac{1}{t}x + \sin t \quad (\text{for } t > 0), \\ x(\pi) = 1. \end{cases} \tag{7.5}$$

The ODE in (7.5) is not separable, but since it can be written in the form $x' + p(t)x = q(t)$, means it may be solved by multiplying by an *integrating factor* of the form $e^{I(t)}$, where

$$I(t) = \int p(t)\, dt.$$

In this case, $e^{\log|t|} = t$. Multiplying both sides of the equation by the integrating factor yields

$$\begin{aligned} tx' + x &= t \sin t \\ (tx)' &= t \sin t. \end{aligned}$$

By integrating both sides with respect to t, we obtain $tx = -t\cos t + \sin t + C$. Thus the general solution is

$$x = -\cos t + \frac{\sin t}{t} + \frac{C}{t},$$

and the specific solution to our initial value problem is

$$x = -\cos t + \frac{\sin t}{t}$$

as $x(\pi) = 1$ implies that $C = 0$. $\square$

7.1 Existence and uniqueness of solutions

We now turn to the central question(s) of whether a system has a solution and if so, whether it is unique or not. Before moving on, we pause to describe *why* we care so much about this.

Suppose we have a system we are trying to understand. We develop a (mathematical) model to help us understand it and predict its behavior. Regardless of our ability to find a solution to our model, one exists in reality, so we want to understand what conditions on our models guarantee that a solution will exist. Once we know solutions exist, we want to make sure we have the correct one that will actually model the system we care about. If we can guarantee that solutions are unique, then we do not have to worry about guessing. Any solution we find will be correct by uniqueness.

Example 7.4. Consider the IVP

$$\begin{cases} x' = 3x^{2/3} \\ x(0) = 0. \end{cases} \tag{7.6}$$

The IVP in (7.6) is separable, and the solution is $x(t) = t^3$. However, we have implicitly used that $x \neq 0$. Consider, for $\alpha \geq 0$,

$$x_\alpha(t) = \begin{cases} (t+\alpha)^3 & \text{when } t < -\alpha, \\ 0 & \text{when } -\alpha \leq t \leq \alpha, \\ (t-\alpha)^3 & \text{when } \alpha < t. \end{cases}$$

A quick check verifies that x_α is a solution to the IVP (7.6) for all $\alpha \geq 0$. See Figure 7.1 for sample plots. The problem has therefore infinitely many solutions. (Now the question is which solution accurately models the system?)

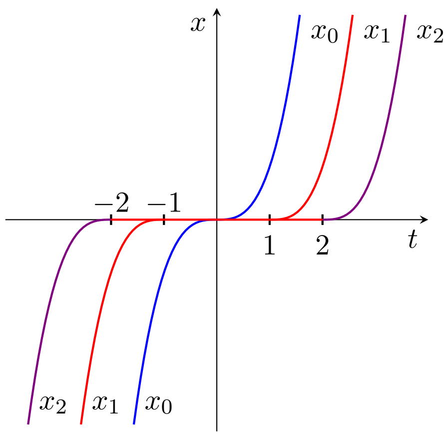


Figure 7.1: Three functions $x_\alpha(t)$ from Example 7.4 are graphed (in red, blue, and purple).

Definition 7.5. A function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is said to satisfy the **Lipschitz condition** on a set $\Omega \subseteq \mathbb{R} \times \mathbb{R}^d$ if there is a constant $L > 0$ such that for all $(t, x), (t, y) \in \Omega$

$$\|f(t,x) - f(t,y)\| \leq L \|x - y\| .$$

The function f is also sometimes referred to as **Lipschitz continuous** on Ω if it satisfies the Lipschitz condition on Ω.

Example 7.6. Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(t, x) = t^2 x^2$. Since for all $x, y, t \in \mathbb{R}$ such that $x, y, t \in B_1(0)$,

$$\begin{aligned} |f(t, x) - f(t, y)| &= |t^2(x^2 - y^2)| \\ &\leq |(x - y)(x + y)| \\ &\leq (|x| + |y|)|x - y| \\ &\leq 2|x - y|, \end{aligned}$$

it follows that on

$$\Omega = \{(t, x) : |t| \leq 1 \text{ and } |x| \leq 1\}$$

the function f satisfies the Lipschitz condition. □

We can think of a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as a one-dimensional function by writing $f(t, x) =: f_t(x)$. Then a one-dimensional function that is differentiable and has bounded derivative will always satisfy the Lipschitz condition. This follows from the MVT in fact as $|f(x) - f(y)| = |f'(\xi)||x - y|$. However, the reverse implication does not hold, and this is shown in the next example.

Example 7.7. Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(t, x) = |x|$. For all $x, y, t \in \mathbb{R}$,

$$|f(t, x) - f(t, y)| = ||x| - |y|| \leq |x - y|,$$

so the function fulfils a Lipschitz condition on $\mathbb{R} \times \mathbb{R}$. But it is not differentiable on any set containing the line $x = 0$. □

All functions $f(t, x)$ that satisfy the Lipschitz condition are continuous in the x variable. The reverse implication, however, is not true.

Example 7.8. Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $f(t, x) = 3x^{2/3}$, which is continuous everywhere. Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ be a neighborhood of 0. Then for $y = 0$ and $(t, x) \in \Omega$,

$$|f(x) - f(y)| = \left|3x^{2/3} - 0\right| = 3\left|x^{2/3}\right| = 3\left|x^{-1/3}\right||x - 0|,$$

and $|x^{-1/3}|$ is not bounded close to $x = 0$. Thus f does not satisfy the Lipschitz condition on any neighborhood of $x = 0$. □

If an IVP is defined by a function satisfying the Lipschitz condition on a bounded set, then the IVP possesses a unique solution on some interval. The next theorem is attributed to Cauchy, Lindelöf, Lipschitz, and Picard.[1]

Theorem 7.9 (Existence and uniqueness). *Suppose that the function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and the*

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0 \end{cases} \tag{7.7}$$

defines an IVP. If there exists $a, b \in \mathbb{R}$ such that f satisfies the Lipschitz condition on

$$\Omega = \{(t, x) \in \mathbb{R} \times \mathbb{R}^d : |t - t_0| \leq a, \|x - x_0\| \leq b\},$$

then there exists $c > 0$ such that the IVP in (7.7) has a unique solution for all t in the interval

$$I_c := \{t \in \mathbb{R} : |t - t_0| \leq c\}.$$

[1] https://en.wikipedia.org/wiki/Picard-Lindelöf_theorem

Note that Theorem 7.9 tells us nothing about the size of the interval on which a unique solution exists.

Example 7.10. Consider the IVP given by

$$\begin{cases} x'(t) = tx^2, \\ x(0) = 1. \end{cases}$$

The function $f(t,x) = tx^2$ satisfies the Lipschitz condition on any bounded set, for example in $\Omega = [-2,2] \times [-1,1]$, so using the variables in Theorem 7.9, $t_0 = x_0 = 0$, $a = 2$, and $b = 1$. Separating the variables and applying the initial condition, a solution for the IVP is given by

$$x(t) = \frac{2}{2 - t^2}.$$

This solution does not exist for all values of t, but it is valid for all $t \in (-\sqrt{2}, \sqrt{2})$, which contains the interval guaranteed by Theorem 7.9. □

The following is a useful corollary to Theorem 7.9.

Corollary 7.11. *If $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous on $\mathbb{R} \times \mathbb{R}^d$, then for every $t_0 \in \mathbb{R}$, the IVP*

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0 \end{cases}$$

has a unique solution for all $x_0 \in \mathbb{R}^d$.

7.2 Fixed points and autonomous systems

As in the discrete case, the set of fixed points of a continuous dynamical system are important to understanding the long-term behavior of the system.

Definition 7.12. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $x' = f(x)$ define an autonomous dynamical system. A point $\bar{x} \in \mathbb{R}^d$ such that $f(\bar{x}) = 0$ is called a **fixed point** of the system.

Notice the difference between the discrete case compared with the continuous case. In the discrete case, we are *iterating* the function: the system $x_{n+1} = f(x_n)$ has a fixed point at $\bar{x}$ if $f(\bar{x}) = \bar{x}$. On the other hand in the continuous case, the function is *defining* the derivative (the change in x with respect to t). Since fixed points do not change with time, the (time) derivative is 0—that is, $f(\bar{x}) = 0$.

As in the discrete case, if a dynamical system (in $\mathbb{R}$) is defined by a continuous function, then the limit of a convergent orbit must be a fixed point of the system.

Proposition 7.13. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $x' = f(x)$ is an autonomous dynamical system with orbit $x(t)$. If there exists $\bar{x} \in \mathbb{R}$ such that*

$$\lim_{t \to \infty} x(t) = \bar{x},$$

then $\bar{x}$ is a fixed point of the system.

Proof. Since f is continuous and $x(t)$ is an orbit,

$$\lim_{t \to \infty} x'(t) = \lim_{t \to \infty} f(x(t)) = f(\bar{x}). \tag{7.8}$$

We show that if $f(\bar{x}) = C$, then $C = 0$, so that $\bar{x}$ is a fixed point.

Suppose, via contradiction, that $C > 0$. Then from (7.8), there exists $a \in \mathbb{R}$ such that for all $t > a$, $x'(t) > C/2$. By the Mean Value Theorem, for all $t > a$ that

$$x(t) - x(a) \geq \frac{C}{2}(t - a),$$

which implies

$$x(t) \geq x(a) + \frac{C}{2}(t - a). \tag{7.9}$$

Since a and C are constants, the right side of (7.9) grows arbitrarily large as $t \to \infty$. This contradicts the fact that the limit exists, so that $C \leq 0$. A similar argument applies in the case when $C < 0$. Therefore, $C = 0$, so $\bar{x}$ is a fixed point. □

Proposition 7.14. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz continuous on $\mathbb{R}^d$, and let $\bar{x}$ be a fixed point of the dynamical system $x' = f(x)$. If an orbit $x(t)$ of the system contains the state $\bar{x}$, then the orbit is constant.*

Proof. Let $x(t)$ be an orbit of the system passing through $\bar{x}$. Since $x(t)$ is an orbit, it is a solution of the system. Thus, by definition for all t and for some $\bar{t}$,

$$\begin{cases} x'(t) = f(x(t)), \\ x(\bar{t}) = \bar{x}. \end{cases}$$

Define a constant function $\widetilde{x} : \mathbb{R} \to \mathbb{R}^d$ by $\widetilde{x}(t) := \bar{x}$ for all $t \in \mathbb{R}$. Then for all $t \in \mathbb{R}$,

$$\widetilde{x}'(t) = 0 = f(\bar{x}) = f(\widetilde{x}(t))$$

and $\widetilde{x}(\bar{t}) = \bar{x}$. Thus, $\widetilde{x}(t)$ is also a solution of the IVP. Because f is Lipschitz continuous, Theorem 7.9 implies that the solution is unique. Hence, for all $t \in \mathbb{R}$,

$$x(t) = \widetilde{x}(t) = \bar{x}.$$ □

The Lipschitz property imposes constraints we can see in the system. Proposition 7.14 ensures that Lipschitz continuous functions cannot have orbits that eventually hit fixed points; either the orbit is fixed or it is not. This is not the only constraint imposed by Lipschitz continuity; we can quickly prove some another property in the one-dimensional case.

Proposition 7.15. *If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}$, then every orbit of the system defined by $x' = f(x)$ is either constant or strictly monotone.*

Proof. Let $x(t)$ be an orbit of the system. If $x(t)$ is not monotone, then there exists a t_0 such that

$$0 = x'(t_0) = f(x(t_0)). \tag{7.10}$$

Equation (7.10) implies that the orbit contains a fixed point. Since f is Lipschitz continuous, by Proposition 7.14, the orbit is constant. □

7.3 Flows

Flows provide a geometric interpretation through which we can study continuous dynamical systems.

Example 7.16. First consider the system given by

$$x' = \sin x. \tag{7.11}$$

The family of solutions to this is

$$t = \log \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|. \tag{7.12}$$

Suppose we want to understand how the system behaves with a given initial condition x_0 as $t \to \infty$. The exact solutions given in (7.12) are challenging to analyze for arbitrary x_0 (even for particularly "nice" choices this is hard). Instead, the ODE in (7.11) is simple to graph. If we interpret t as time, x as the position of a particle on $\mathbb{R}$, and x' as velocity of that particle, then $x' = \sin x$ represents a *vector field*. Vectors point to the right when $x' > 0$ and to the left when $x' < 0$. This is plotted in Figure 7.2. □

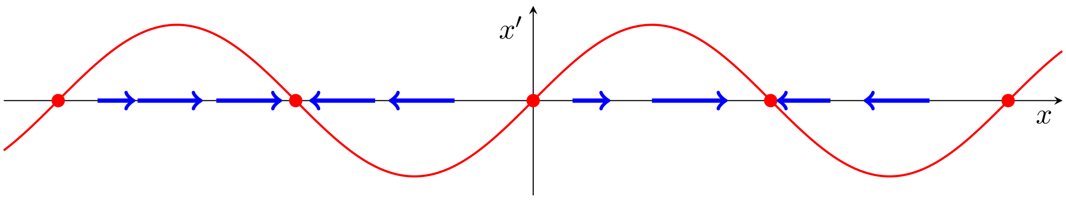


Figure 7.2: A plot of flows (in blue) on the real line from Example 7.16.

The plot in Figure 7.2 illustrates which fixed points should be considered stable and which should be unstable. This is one of the main benefits of flows, but before defining anything, we pick up with another example.

Example 7.17. For some fixed $x_0 \in \mathbb{R}$, consider the initial value problem

$$\begin{cases} x' = x, \\ x(0) = x_0. \end{cases}$$

The IVP has solution given by $x(t) = x_0 e^t$. Now define a function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\Phi(t, x_0) := x_0 e^t,$$

for $(t, x_0) \in \mathbb{R} \times \mathbb{R}$. The function Φ is continuous and satisfies

(i) for all $x_0 \in \mathbb{R}$,

$$\Phi(0, x_0) = x_0 e^0 = x_0,$$

(ii) for all $x_0, s, t \in \mathbb{R}$,

$$\Phi(s + t, x_0) = x_0 e^{s+t} = x_0 e^s e^t = \Phi(t, x_0) e^s = \Phi(s, \Phi(t, x_0)).$$

This function Φ is an example of a global flow. We may drop the subscript from x_0 in the notation $\Phi(t, x_0)$ as becomes more cumbersome, but the second parameter should be interpreted as the initial condition.

Definition 7.18. Let $M \subseteq \mathbb{R}^d$. A continuous map $\Phi : \mathbb{R} \times M \to M$ is called a **flow** if

(i) $\Phi(0, x) = x$ for all $x \in M$,

(ii) $\Phi(s + t, x) = \Phi(s, \Phi(t, x))$ for all $s, t \in \mathbb{R}$ and $x \in M$.

Given an autonomous dynamical system defined by a Lipschitz function, we can use the solution to define a flow, as we did in Example 7.17. For an initial state x of the system, the flow $\Phi(t, x)$ gives the state of the system at time t. The following definition of orbit for a flow agrees with our earlier definition for dynamical systems.

Definition 7.19. Let $M \subseteq \mathbb{R}^d$ and $\Phi : \mathbb{R} \times M \to M$ be a flow. For $x_0 \in M$,

(i) $x(t) = \Phi(t, x_0)$ is called the **orbit** through x_0.

(ii) The set $T(x_0) = T_\Phi(x_0) = \{\Phi(t, x_0) : t \in \mathbb{R}\}$ is called the **trajectory** through x_0.

The trajectory through a given point x_0 is the *set of states* taken by the system when starting at x_0. The orbit through x_0 is the *function* that describes the trajectory. Using the flows from Example 7.17, the orbit through 2 is the function $x(t) = 2e^t$, and the trajectory is $T(2) = (0, \infty)$.

Definition 7.20. Let $M \subseteq \mathbb{R}^d$ and $\Phi : \mathbb{R} \times M \to M$ be a flow.

(i) A point $\bar{x} \in M$ is a **fixed point** of the flow if $\Phi(t, \bar{x}) = \bar{x}$ for all $t \in \mathbb{R}$.

(ii) A point $\bar{x} \in M$ is a **periodic point** if there is a $\bar{t} > 0$ such that $\Phi(\bar{t}, \bar{x}) = \bar{x}$. The smallest such $\bar{t} > 0$ is the **period** of $\bar{x}$;

We verify that this definition of fixed points matches previous one in Section 7.2. If $\bar{x} \in \mathbb{R}$ is a fixed point, then

$$\frac{d\Phi}{dt}(t, \bar{x}) = \frac{d\bar{x}}{dt} = 0.$$

Therefore, this matches the definitions where $f(\bar{x}) = 0$ when $x' = f(x)$. The trajectory of $\bar{x}$ is exactly what we expect: $T(\bar{x}) = \{\bar{x}\}$. On the other hand, if $\bar{x}$ is a periodic point, then there exists $\bar{t} > 0$ such that $\Phi(\bar{t}, \bar{x}) = \bar{x}$. From the properties satisfied by all flows: for all $n \in \mathbb{Z}$,

$$\Phi(\bar{t}, \bar{x}) = \Phi(n\bar{t}, \bar{x}).$$

Like in the discrete case, the orbit of $\bar{x}$ is also said to be periodic.

Definition 7.21. Let $M \subseteq \mathbb{R}^d$ and $\Phi : \mathbb{R} \times M \to M$ be a flow. Assume that $\bar{x} \in M$ be a fixed point.

(i) The point $\bar{x}$ is a **Lyapunov stable** fixed point if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\bar{x} - y\| < \delta$ implies that for all $t > 0$,

$$\|\Phi(t, \bar{x}) - \Phi(t, y)\| < \varepsilon.$$

(ii) The point $\bar{x}$ is **asymptotically stable** if $\bar{x}$ is stable and there exists $\delta > 0$ such that $\|\bar{x} - y\| < \delta$ implies

$$\lim_{t \to \infty} \Phi(t, y) = \bar{x}.$$

Example 7.22. (*Asymptotically stable fixed point*) Fix $x_0 \in \mathbb{R}$, and consider the IVP

$$\begin{cases} x' = -x, \\ x(0) = x_0. \end{cases}$$

This example is similar to Example 7.17, so this IVP has the solution

$$x(t) = x_0 e^{-t} = \Phi(t, x_0).$$

The point $x_0 = 0$ is a fixed point of the system since $\frac{d\Phi}{dt}(t, 0) = 0$. Thus, for $y \in \mathbb{R}$ and $t \geq 0$,

$$|\Phi(t, 0) - \Phi(t, y)| = |0e^{-t} - ye^{-t}| = e^{-t}|0 - y| = e^{-t}|y| \leq e^0|y| = |y|.$$

Thus, for every $\varepsilon > 0$, choose $\delta = \varepsilon$ so that $|y| = |0 - y| < \delta$ implies that

$$|\Phi(t, 0) - \Phi(t, y)| \leq |y| < \delta = \varepsilon.$$

Thus the fixed point $x = 0$ is Lyapunov stable. Further, for any y, we have that $\Phi(t, y) = ye^{-t} \to 0$ as $t \to \infty$, so the fixed point $x = 0$ is asymptotically stable. $\square$

Chapter 8

Linear ODEs of higher dimension

A linear ODE in $\mathbb{R}^d$ may be written in the form $x'(t) = Ax(t)$, where $A = (a_{ij})$ is a $d \times d$ matrix with entries in $\mathbb{R}$ and $x(t) = (x_1(t), \dots, x_d(t))^\top$. With this, the system

$$\begin{array}{ccccccc} x_1' & = & a_{11}x_1 & + & \cdots & + & a_{1n}x_n, \\ & \vdots & & & & & \\ x_n' & = & a_{n1}x_1 & + & \cdots & + & a_{nn}x_n, \end{array}$$

can be expressed as $x' = Ax$. If this were a 1-dimensional system, say $A = \big(a\big)$, then we could write the solution as the exponential function $x(t) = Ce^{at}$.

8.1 Matrix exponentials

This is not unique to 1-dimensional systems; the idea is to "exponentiate" a matrix: e^A. So what does this mean? Recall the Taylor series of the (real-valued) exponential function $f(x) = e^x$

$$e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m. \tag{8.1}$$

The right side of (8.1) is a *power series*, which converges everywhere (i.e., it converges for all $x \in \mathbb{R}$). Although we cannot exponentiate a matrix, per se, we can consider a power series (and its convergence properties) evaluated at matrix.

Definition 8.1. For $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, the **exponential** of A is the $d \times d$ matrix, denoted by e^A,

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!}. \tag{8.2}$$

We furthermore define $A^0 = I$ for all $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, and when $A = 0$ is the zero matrix, we set $e^A = I_d$.

An important issue concerns the convergence of (8.2). It is not immediately clear that the limit of the partial sums of the matrices converges to a matrix. Using the convergence analysis of the Taylor series (8.1) and a property of the matrix norm, one can show that the sum in (8.2) converges for all matrices $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$. The property that does this is the *submultiplicativity* property. For $A, B \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, $\|AB\| \leq \|A\| \, \|B\|$, and hence for $n \in \mathbb{N}$, $\|A^n\| \leq \|A\|^n$.[1] If $(e^A)_{ij}$ is the (i, j) entry of the matrix e^A, then

$$|(e^A)_{ij}| \leq \sum_{m=0}^{\infty} \left| \frac{(A^m)_{ij}}{m!} \right| \leq \sum_{m=0}^{\infty} \frac{1}{m!} \|A^m\| \leq \sum_{m=0}^{\infty} \frac{1}{m!} \|A\|^m,$$

which is finite by the convergence of the power series in 8.1.

[1] Not every matrix norm has this property.

Example 8.2. If A is a diagonal matrix, then its exponential is easy to calculate. For

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{dd} \end{pmatrix},$$

the mth power is

$$A = \begin{pmatrix} a_{11}^m & 0 & \dots & 0 \\ 0 & a_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{dd}^m \end{pmatrix}.$$

Thus, its exponential is

$$\begin{aligned} e^A &= \sum_{m=0}^{\infty} \frac{A^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \begin{pmatrix} a_{11}^m & 0 & \dots & 0 \\ 0 & a_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{dd}^m \end{pmatrix} \\ &= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{a_{11}^m}{m!} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{m=0}^{\infty} \frac{a_{dd}^m}{m!} \end{pmatrix} \\ &= \begin{pmatrix} e^{a_{11}} & 0 & \dots & 0 \\ 0 & e^{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_{dd}} \end{pmatrix}. \end{aligned}$$

Therefore, the exponential of a diagonal matrix is the diagonal matrix of the exponentials of each of the diagonal entries. □

Matrices $A, B \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ are said to commute if $AB = BA$.

Proposition 8.3. *Suppose $A, B, S \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ where S is invertible. If A and B are commuting, then*

1. $e^{A+B} = e^A e^B$,
2. $(e^A)^{-1} = e^{-A}$,
3. $(e^A)^m = e^{mA}$ *for* $m \in \mathbb{Z}$,
4. $e^{S^{-1}AS} = S^{-1}e^A S$.

Like with vectors, we consider matrices with entries that are functions of some parameter t. For example, $x(t) = (\sin t, \sqrt{t})^\top$ and

$$A(t) = \begin{pmatrix} t & t^2 \\ 1 & \sin t \end{pmatrix}.$$

Therefore, we can differentiate with respect to t entry-wise:

$$\frac{dA}{dt} = \begin{pmatrix} 1 & 2t \\ 0 & \cos t \end{pmatrix}.$$

Proposition 8.4. *For $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$,*

$$\frac{d}{dt} e^{tA} = Ae^{tA}.$$

Proof. By the definition of the exponential matrix,

$$\begin{aligned}
\frac{d}{dt} e^{tA} &= \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(tA)^m}{m!} && \text{(def. of exponential)} \\
&= \sum_{m=1}^{\infty} \frac{mt^{m-1}A^m}{m!} && \text{(chain rule)} \\
&= A \sum_{m=1}^{\infty} \frac{t^{m-1}A^{m-1}}{(m-1)!} && \text{(rearranging)} \\
&= Ae^{tA}. && \text{(def. of exponential)}
\end{aligned}$$

□

Proposition 8.4 allows us to express solutions of linear differential equations in terms of matrix exponentials. However, before we see exactly how this works, we first look at more techniques that will help us calculate e^A for a general matrix A.

Definition 8.5. A matrix $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ is said to be **diagonalizable** if there is an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$.

Not all matrices are diagonalizable over the reals or complex numbers. For example, the two matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are not diagonalizable over $\mathbb{R}$. Over $\mathbb{C}$, the right matrix is diagonalizable. In particular, a $d \times d$ matrix with d distinct eigenvalues is diagonalizable.

Proposition 8.6. *A matrix $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ is diagonalizable if and only if A has d linearly independent eigenvectors, $\{v_1, v_2, \ldots, v_d\}$. In this case, the matrix S with columns $v_1, v_2, \ldots, v_d$ and the diagonal matrix D with diagonal entries given by corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ satisfy $S^{-1}AS = D$.*

Now we see how Proposition 8.6 can help us compute matrix exponentials. Suppose A is diagonalizable. Then by Proposition 8.6, there exists an invertible matrix S, whose columns are eigenvectors of A, such that $S^{-1}AS = D$ for some diagonal matrix D. This implies that $A = SDS^{-1}$. Therefore,

$$\begin{aligned}
e^A &= e^{SDS^{-1}} && \text{(diagonalizable)} \\
&= Se^DS^{-1} && \text{(Proposition 8.3)} \\
&= S \begin{pmatrix} e^{D_{11}} & & \\ & \ddots & \\ & & e^{D_{dd}} \end{pmatrix} S^{-1}. && \text{(diagonal exponential)}
\end{aligned}$$

Example 8.7. Let

$$A = \begin{pmatrix} -4 & 6 & -3 \\ 0 & 2 & 0 \\ 6 & -6 & 5 \end{pmatrix}.$$

The eigenvalues of A are -1, 2 and 2, with corresponding eigenvectors

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

To diagonalize A, we set

$$S = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix},$$

so that

$$S^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Here, $D = \operatorname{diag}(-1, 2, 2)$. Now we can compute the exponential of A:

$$\begin{aligned} e^A &= Se^D S^{-1} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} -2 & 2 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-1} - e^2 & -2e^{-1} + 2e^2 & e^{-1} - e^2 \\ 0 & e^2 & 0 \\ -2e^{-1} + 2e^2 & 2e^{-1} - 2e^2 & -e^{-1} + 2e^2 \end{pmatrix} \\ &\approx \begin{pmatrix} -6.65 & 14.04 & -7.02 \\ 0.00 & 7.39 & 0.00 \\ 14.04 & -14.04 & 14.41 \end{pmatrix}. \ \square \end{aligned}$$

There is a family of matrices, different from diagonal matrices, where computing the (matrix) exponential involves only finite sums. These are matrices where the sequence $(A^m)_{m=0}^{\infty}$ eventually stabilizes at the zero matrix.

Definition 8.8. A matrix $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ is said to be **nilpotent** if there exists $n \in \mathbb{N}$ such that A^n is the zero matrix.

We will denote the zero matrix by 0, so the context should make it clear. If $A^n = 0$, then $A^k = 0$ for all $k \geq n$, since

$$A^k = A^n A^{k-n} = 0A^{k-n} = 0.$$

And as described above, if A is nilpotent, then the sum in the definition of e^A is finite. That is, if $A^n = 0$, then

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} = \sum_{m=0}^{n-1} \frac{A^m}{m!}.$$

Example 8.9. Let

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can rewrite A as a sum of two matrices in the following way

$$A = I + N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix N is nilpotent since $N^3 = 0$. Thus, since $e^I = eI$,

$$\begin{aligned} e^A = e^{I+N} = e^I e^N &= eI \left(I + N + \frac{1}{2}N^2 \right) \\ &= e \left(A + \frac{1}{2}N^2 \right) \\ &= e \left(\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\ &= e \begin{pmatrix} 1 & 1 & 9/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \square \end{aligned}$$

8.2 Characteristic polynomials

Recall that the **characteristic polynomial** p_A of a matrix A is given by

$$p_A(\lambda) = \det(\lambda I - A).$$

If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, then p_A is a polynomial of degree d, and the eigenvalues of A are the values of λ that satisfy the equation $p_A(\lambda) = 0$. The equation $p_A(\lambda) = 0$ is sometimes called the **characteristic equation** of A. In general, the characteristic equation has the form, for $c_i \in \mathbb{R}$,

$$\lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1\lambda + c_0 = 0.$$

As we know how to take powers, scalar multiples, and sums of matrices, we can also easily form polynomials of matrices. Note that the constant term of a polynomial $p(\lambda)$ can be written as $c_0\lambda^0$, so the corresponding term of a matrix polynomial $p(A)$ is $c_0A^0 = c_0I$. In particular, we can evaluate the characteristic polynomial $p_A(\lambda)$ of a matrix A at A.

Example 8.10. The characteristic polynomial of the matrix A from Example 8.9 is

$$p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & -4 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3.$$

Plugging the matrix A into its own characteristic polynomial gives

$$p_A(A) = (A - I)^3 = N^3 = 0. \square$$

This result from Example 8.10 not just a coincidence; it is the result of an important theorem in linear algebra.[2]

Theorem 8.11 (Cayley–Hamilton Theorem)**.** *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{C})$, then $p_A(A) = 0$.*

Note that if A is a real square matrix, then by default it is also a complex matrix and satisfies the hypothesis of the Cayley–Hamilton Theorem. We will not immediately see how to apply the Cayley–Hamilton Theorem to our situation, but we will soon. Because eigenvalues of real-valued matrices can be complex, we need to discuss some properties of complex-valued functions.

[2] See the page on the Cayley–Hamilton Theorem and its applications.

Definition 8.12. An **entire function** $f : \mathbb{C} \to \mathbb{C}$ is a function that can be expressed as a power series with an infinite radius of convergence. That is, there exists a complex sequence $(a_k)_{k=0}^{\infty}$ such that for all $z \in \mathbb{C}$,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

It follows that $f(z) = e^z$ is an entire function, where $a_k = 1/k!$. The following proposition, which is reminiscent of the Euclidean algorithm, will allow us to calculate exponentials of matrices using polynomials.

Proposition 8.13. *Let f be an entire function, and let p be a polynomial of degree n. Then there exists an entire function g and a polynomial q of degree $\leq n-1$ such that*

$$f(z) = g(z)p(z) + q(z).$$

Proof. We prove this by induction on the degree n of q. For the base case $n = 1$, suppose $p(z) = z - c$. If $c = 0$, then we are done since

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \left(\sum_{k=1}^{\infty} a_k z^{k-1} \right) z + a_0 = \left(\sum_{k=0}^{\infty} a_{k+1} z^k \right) z + a_0.$$

If $c \neq 0$, then we need to prove that there exists an entire function g and a constant q_0 such that $f(z) = g(z)(z-c) + q_0$. Setting $w = z - c$, this becomes

$$f(w+c) = g(w+c)w + q_0.$$

This follows directly from the case when $c = 0$ since $f(w+c)$ and $g(w+c)$ are both entire functions. This proves the base case.

For induction step, assume the proposition holds for polynomials of degree $\leq n-1$. Let p_1 be a polynomial of degree $n-1$, and for some $c \in \mathbb{C}$, let $p(z) = (z-c)p_1(z)$ for all $z \in \mathbb{C}$. Thus, p is a polynomial of degree n. By the induction assumption, there exists g_1 and q_1 such that

$$f(z) = g_1(z)p_1(z) + q_1(z)$$

where q_1 is a polynomial of degree $\leq n-2$. Furthermore, as g_1 is an entire function, we can write $g_1(z) = g(z)(z-c) + q_0$, where q_0 is a constant. Therefore,

$$\begin{aligned}
f(z) &= g_1(z)p_1(z) + q_1(z) \\
&= (g(z)(z-c) + q_0)\, p_1(z) + q_1(z) \\
&= g(z)(z-c)p_1(z) + q_0 p_1(z) + q_1(z) \\
&= g(z)p(z) + q_0 p_1(z) + q_1(z).
\end{aligned}$$

Now set $q(z) = q_0 p_1(z) + q_1(z)$, which is a polynomial with degree at most $n-1$. $\square$

Let $A \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ with characteristic polynomial p_A, and let f be an entire function. By Proposition 8.13, there is a polynomial q and an entire function g such that for all $z \in \mathbb{C}$,

$$f(z) = g(z)p_A(z) + q(z).$$

By the Cayley–Hamilton Theorem,

$$f(A) = g(A)p_A(A) + q(A) = q(A),$$

so we can easily calculate $f(A)$ if we know the polynomial q.

Proposition 8.14. *Let $A \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ and corresponding multiplicities $n_1, \ldots, n_m$. If f is an entire function, then there exists a unique polynomial q of degree $\leq d-1$ defined by the conditions for all $k \in \{1, \ldots, m\}$ and $j \in \{0, \ldots, n_k - 1\}$*

$$\frac{d^j q}{dz^j}(\lambda_k) = \frac{d^j f}{dz^j}(\lambda_k), \tag{8.3}$$

such that $f(A) = q(A)$.

Proof. By Proposition8.13, there is a polynomial q of degree at most $d-1$ such that

$$f(z) = g(z)p_A(z) + q(z).$$

By the Cayley–Hamilton Theorem, we have $f(A) = q(A)$. From the assumptions on eigenvalues of A, it follows that

$$p_A(z) = \prod_{k=1}^{m} (z - \lambda_k)^{n_k}.$$

Since

$$f(z) - q(z) = g(z) \prod_{k=1}^{m} (z - \lambda_k)^{n_k},$$

for each $k \in \{1, \ldots, m\}$, both f and q have the same derivatives of order $1, 2, \ldots, n_k - 1$ at $z = \lambda_k$. That is, the conditions in (8.3) holds. □

Using this, we may find a polynomial to calculate the exponential of a matrix.

Example 8.15. The matrix

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$ (and thus, both have multiplicity 1). As A has dimension 2, we look for a polynomial q of degree 1 such that $e^A = q(A)$. Let $q(z) = az + b$. Then, by Proposition 8.14, we require that

$$q(-1) = e^{-1}, \qquad\qquad q(-3) = e^{-3}.$$

This gives the following linear system

$$\begin{array}{rcrcl} -a & + & b & = & e^{-1}, \\ -3a & + & b & = & e^{-3}. \end{array}$$

Therefore,

$$\begin{aligned} a &= \frac{1}{2}\left(e^{-1} - e^{-3}\right) \\ b &= \frac{1}{2}\left(3e^{-1} - e^{-3}\right). \end{aligned}$$

Now we can explicitly write the exponential.

$$\begin{aligned} e^A &= q(A) = aA + bI \\ &= \frac{1}{2}\left(e^{-1} - e^{-3}\right)\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \frac{1}{2}\left(3e^{-1} - e^{-3}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{2}e^{-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}e^{-3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \;\square \end{aligned}$$

Example 8.16. We calculate again the exponential of the matrix A from Example 8.9 and Example 8.10, where

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

As the characteristic polynomial of A is $p_A(\lambda) = (\lambda - 1)^3$, we see that A has one eigenvalue, $\lambda_1 = 1$, with multiplicity 3. From Proposition 8.14, we are looking for quadratic polynomial, $q(z) = az^2 + bz + c$, such that

$$\begin{cases} q(1) = f(1) \\ q'(1) = f'(1) \\ q''(1) = f''(1) \end{cases} \Leftrightarrow \begin{cases} a + b + c = e^1 \\ 2a + b = e^1 \\ 2a = e^1 \end{cases} \Leftrightarrow \begin{cases} a = \frac{e}{2} \\ b = 0 \\ c = \frac{e}{2}. \end{cases}$$

Thus, $q(z) = \frac{e}{2}z^2 + \frac{e}{2}$. Solving for the exponential,

$$\begin{aligned} e^A = q(A) &= \frac{e}{2}A^2 + \frac{e}{2}I \\ &= \frac{e}{2}\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 + \frac{e}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{e}{2}\begin{pmatrix} 1 & 2 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} + \frac{e}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e\begin{pmatrix} 1 & 1 & 9/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \square \end{aligned}$$

8.3 Solving linear ODEs

We are finally ready to apply these results to solving linear ODEs.

Proposition 8.17. *Let $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$. The unique general solution to the IVP*

$$\begin{cases} x'(t) = Ax(t), \\ x(t_0) = x_0 \end{cases} \tag{8.4}$$

is given by

$$x(t) = e^{A(t-t_0)}x_0.$$

Proof. By Proposition 8.4,

$$x'(t) = \frac{d}{dt}\left(e^{A(t-t_0)}x_0\right)' = Ae^{A(t-t_0)}x_0 = A \cdot x(t).$$

Thus, $x(t)$ satisfies the differential equation. For the initial condition in (8.4),

$$x(t_0) = e^{A(t_0-t_0)}x_0 = e^0 x_0 = Ix_0 = x_0.$$

To prove uniqueness, we show that $f(t, x) = Ax$ satisfies the Lipschitz condition on $\mathbb{R} \times \mathbb{R}^d$. For all $t \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^d$,

$$\|f(t, x) - f(t, y)\| = \|Ax - Ay\| \leq \|A\| \|x - y\|.$$

By Theorem 7.9, uniqueness follows. $\square$

The solutions to the system (8.4) define a flow. To see this, let $\Phi(t, x_0) = e^{At}x_0$. Then $\Phi(0, x_0) = e^{A0}x_0 = x_0$, and

$$\Phi(s + t, x_0) = e^{A(s+t)}x_0 = e^{At}e^{As}x_0 = \Phi(s, e^{At}x_0) = \Phi(s, \Phi(t, x_0)).$$

The fixed points of the system (8.4) are $x = 0$ and the eigenvectors of A that correspond to the eigenvalue 0.

Example 8.18. Consider the IVP

$$\begin{array}{rcrcr} x_1' & = & x_1 & + & 2x_2, \\ x_2' & = & 2x_1 & + & 4x_2, \end{array}$$

where $x(1) = (3, -1)^\top$. Writing this in matrix form, $x' = Ax$, we have

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

To calculate the solution from Proposition 8.17, $x(t) = e^{A(t-t_0)}x_0$, we first find e^{tA}. The matrix A has the eigenvalues 0 and 5 with corresponding eigenvectors

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Let S be the matrix with columns given by the eigenvectors. Since $A = SDS^{-1}$,

$$tA = StDS^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 5t \end{pmatrix} \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Thus,

$$\begin{aligned} e^{tA} = Se^{tD}S^{-1} &= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{5t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} e^{5t}+4 & 2e^{5t}-2 \\ 2e^{5t}-2 & 4e^{5t}+1 \end{pmatrix}. \end{aligned}$$

The solution is thus

$$\begin{aligned} x(t) = e^{A(t-t_0)}x_0 = e^{(t-1)A} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} e^{5(t-1)}+4 & 2e^{5(t-1)}-2 \\ 2e^{5(t-1)}-2 & 4e^{5(t-1)}+1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} e^{5(t-1)}+14 \\ 2e^{5(t-1)}-7 \end{pmatrix}. \end{aligned}$$

As noted above, the fixed points of $x' = Ax$ are either $x = 0$ or the eigenvectors of A corresponding to the eigenvalue 0. Let $x_0 = (-2, 1)^\top$, and let $c \in \mathbb{R}$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned} \Phi(t, cx_0) &= e^{At}cx_0 \\ &= \frac{1}{5} \begin{pmatrix} e^{5t}+4 & 2e^{5t}-2 \\ 2e^{5t}-2 & 4e^{5t}+1 \end{pmatrix} c \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ &= \frac{c}{5} \begin{pmatrix} -10 \\ 5 \end{pmatrix} \\ &= c \begin{pmatrix} -2 \\ 1 \end{pmatrix} = cx_0. \end{aligned}$$

Therefore, the entire subspace spanned by $(-2, 1)^\top$ is fixed by Φ. Flows for this example can be seen in Figure 8.1. $\square$

Theorem 8.19. *If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ has d linearly independent eigenvectors, $v_1, \dots, v_d$, with corresponding eigenvalues $\lambda_1, \dots, \lambda_d$, then there exist constants $c_1, \dots, c_d$, such that the solution to $x' = Ax$ is*

$$x(t) = c_1e^{\lambda_1 t}v_1 + \cdots + c_de^{\lambda_d t}v_d. \tag{8.5}$$

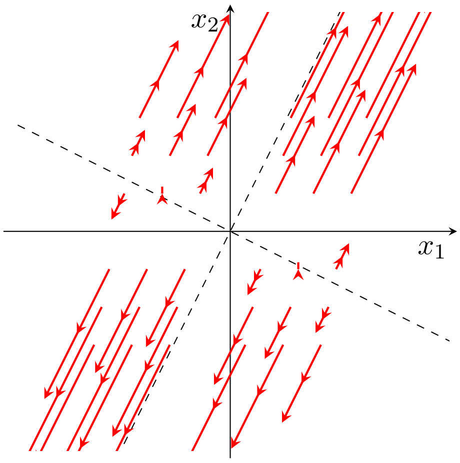


Figure 8.1: A few trajectories of points from Example 8.18

Proof. By Proposition 8.17, the general solution is $x(t) = e^{At}u$, for some $u \in \mathbb{R}^d$. By Proposition 8.6, A is diagonalizable, so there exists $S, D \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ such that $A = SDS^{-1}$, where $S = \begin{pmatrix} v_1 & v_2 & \cdots & v_d \end{pmatrix}$, the matrix whose columns are the eigenvectors v_i. The solution to the ODE is then

$$x(t) = Se^{tD}S^{-1}u.$$

Let

$$S^{-1}u = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix}.$$

Then

$$\begin{aligned} x(t) &= \begin{pmatrix} v_1 & v_2 & \cdots & v_d \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & & \cdots & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & \cdots & e^{\lambda_d t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} \\ &= \begin{pmatrix} v_1 & v_2 & \dots & v_d \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_d e^{\lambda_d t} \end{pmatrix} \\ &= v_1 c_1 e^{\lambda_1 t} + v_2 c_2 e^{\lambda_2 t} + \ldots + v_d c_d e^{\lambda_d t}. \end{aligned}$$

□

Theorem 8.19 gives an alternative method of solving an IVP.

Example 8.20. Consider again the IVP given in Example 8.18. The two eigenvectors of the matrix A are linearly independent, and so by Theorem 8.19, the solution is

$$\begin{aligned} x(t) &= c_1 e^{0t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \end{aligned}$$

for some $c_1, c_2 \in \mathbb{R}$. We can use the initial condition $x(1) = (3, -1)^\top$ to find c_1 and c_2. □

Remark 8.21. If A is not diagonalizable, then terms of the form $t^j e^{\lambda t}$ appear in the solution (8.4).

Theorem 8.22. *The fixed point $x = 0$ of the system $x' = Ax$ is asymptotically stable if and only if* $\mathrm{Re}(\lambda) < 0$ *for all eigenvalues of λ of A.*

Sketch of Proof. If A is diagonalizable, then from Theorem8.19, solutions have the form

$$x(t) = c_1 e^{\lambda_1 t} v_1 + \ldots + c_d e^{\lambda_d t} v_d.$$

Observing that $|e^{\lambda t}| = e^{\mathrm{Re}(\lambda)t}$, it follows that $|e^{\lambda t}|$ approaches 0 if and only if $\mathrm{Re}(\lambda) < 0$. In this case, $\|x(t)\| \to 0$ as $t \to \infty$. In the general case (not diagonalizable), the solution contains coefficients of the form $t^j e^{\lambda t}$, but the absolute value of such terms also tends to 0 as $t \to \infty$ if $\lambda < 0$. □

We now consider in more detail the solutions of 2-dimensional linear ODEs $x' = Ax$.

Example 8.23. We first consider a simple case

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. From before, we know that

$$e^A = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

Thus, the solution to $x'(t) = Ax(t)$ is

$$x(t) = e^{At} x_0 = \begin{pmatrix} e^{\lambda_1 t_0} & 0 \\ 0 & e^{\lambda_2 t_0} \end{pmatrix} \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} x_0^{(1)} \\ e^{\lambda_2 t} x_0^{(2)} \end{pmatrix}.$$

The behavior of the system depends on the values of the eigenvalues λ_1 and λ_2. Some trajectories are graphed in Figure 8.2, but we will analyze the system based on the eigenvalues.

Case 1: $\lambda_1 < 0, \lambda_2 < 0$. The fixed point $x = 0$ is stable and asymptotically stable.

Case 2: $\lambda_1 > 0, \lambda_2 > 0$. The fixed point $x = 0$ is unstable and repelling.

Case 3: $\lambda_1 > 0, \lambda_2 < 0$. The fixed point $x = 0$ is a saddle point. The x-axis is unstable manifold, and the y-axis is stable manifold.

Case 4: $\lambda_1 = 0, \lambda_2 > 0$. There is a fixed line at the x_1-axis, and each point on this line is unstable and repelling.

Case 5: $\lambda_1 = 0, \lambda_2 < 0$. There is a fixed line at the x_1-axis, and each point on this line is stable and asymptotically stable. □

Example 8.24 (Two linearly independent eigenvectors)**.** Consider the system $x' = Ax$ where

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}.$$

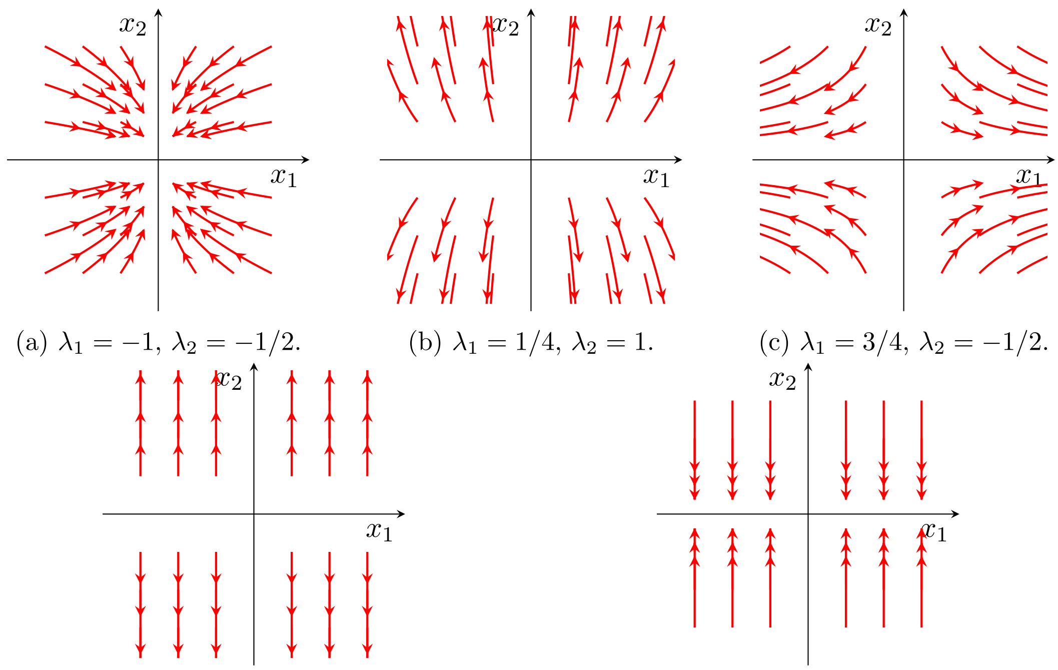
The eigenvalues of A are 2 and -1 with the corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since the eigenvectors are linearly independent, by Theorem 8.19, the solution is

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The phase portrait is plotted in Figure 8.3. □



(a) $\lambda_1 = -1$, $\lambda_2 = -1/2$.

(b) $\lambda_1 = 1/4$, $\lambda_2 = 1$.

(c) $\lambda_1 = 3/4$, $\lambda_2 = -1/2$.

(d) $\lambda_1 = 0$, $\lambda_2 = 1$.

(e) $\lambda_1 = 0$, $\lambda_2 = -1$.

Figure 8.2: Five different cases for a system determined by a diagonal matrix, as discussed in Example 8.23.

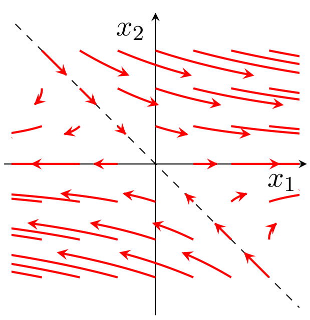


Figure 8.3: The phase portrait of the solution from Example 8.24. The first eigenvector spans the x_1-axis, and the second eigenvector spans the dashed line given by $x_2 = -x_1$.

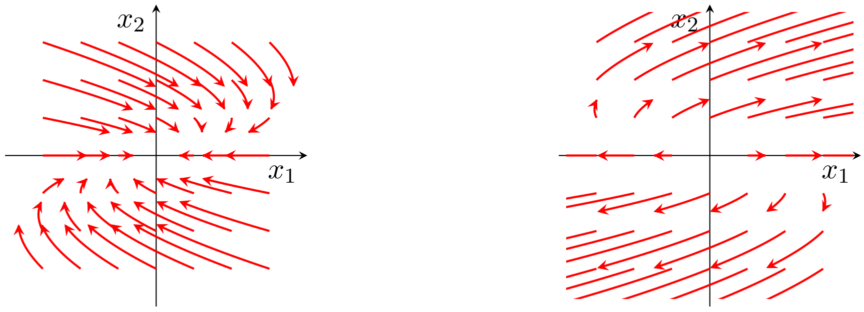


Figure 8.4: Two cases discussed in Example 8.25.

Example 8.25 (One linearly independent eigenvector)**.** Now we consider the matrix

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

which has eigenvalue λ and eigenvector $(1, 0)^\top$ and is thus not diagonalizable. We apply Proposition 8.14 to calculate e^{tA}. Let $f(z) = e^{tz}$ and $q(z) = az + b$. Then we have the system of equations

$$\left\{ \begin{array}{rcl} f(\lambda) & = & q(\lambda), \\ f'(\lambda) & = & q'(\lambda), \end{array} \right. \qquad \Leftrightarrow \qquad \left\{ \begin{array}{rcr} e^{t\lambda} & = & a\lambda + b, \\ te^{t\lambda} & = & a. \end{array} \right.$$

Therefore, $a = te^{\lambda t}$ and $b = e^{\lambda t} - \lambda te^{\lambda t}$, so

$$q(z) = te^{\lambda t} z + e^{\lambda t} - \lambda te^{\lambda t}.$$

Thus

$$e^{tA} = te^{\lambda t} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \left(e^{\lambda t} - \lambda te^{\lambda t} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

So the solutions have the form:

$$x(t) = e^{tA} x_0 = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix}.$$

We can see how the solutions change depending on the sign of the eigenvalue. We know when $\lambda < 0$, $x = 0$ will be stable and attracting, and when $\lambda > 0$, $x = 0$ will be unstable and repelling. This is seen in Figure 8.4. $\square$

Example 8.26 (Complex eigenvalues)**.** If we have a matrix with two complex eigenvalues, we can determine the behavior of the system from the real Jordan normal form

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha I + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \alpha I + \beta J,$$

where $\alpha, \beta \in \mathbb{R}$. The matrices αI and βJ commute, so that

$$e^A = e^{\alpha I + \beta J} = e^{\alpha I} e^{\beta J}.$$

Recall that $e^{\alpha I} = e^\alpha I$. To calculate $e^{\beta J}$, observe that

$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

Therefore, $J^4 = I$. Thus for all $n \in \mathbb{N}$ we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2n} = (-1)^n I, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2n+1} = (-1)^n J.$$

Then

$$\begin{aligned} e^{\beta J} &= \sum_{m=0}^{\infty} \frac{\beta^m J^m}{m!} \\ &= \underbrace{\left(\sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} (-1)^k \right)}_{\cos\beta} I + \underbrace{\left(\sum_{k=0}^{\infty} \frac{\beta^{2k+1}}{(2k+1)!} (-1)^k \right)}_{\sin\beta} J \\ &= (\cos\beta) I + (\sin\beta) J \\ &= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}. \end{aligned}$$

Therefore, the matrix exponential is

$$e^{At} = e^{\alpha t I} e^{\beta t B} = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}.$$

Alternatively, to find e^{tA}, let $f(z) = e^{tz}$ and $q(z) = az + b$. We know that the eigenvalues of A are $\lambda = \alpha \pm i\beta$, so we have the system of equations

$$\begin{cases} f(\alpha + i\beta) = q(\alpha + i\beta) \\ f(\alpha - i\beta) = q(\alpha - i\beta) \end{cases} \Leftrightarrow \begin{cases} e^{\alpha t + i\beta t} = a\alpha + ai\beta + b \\ e^{\alpha t - i\beta t} = a\alpha - ai\beta + b \end{cases}$$

Thus,

$$a = \frac{e^{\alpha t}}{2i\beta} \left(e^{i\beta t} - e^{-i\beta t} \right), \qquad b = \frac{e^{\alpha t}}{2i\beta} \left(e^{i\beta t}(-\alpha + i\beta) + e^{-i\beta t}(\alpha + i\beta) \right).$$

As before, the solution is

$$e^{tA} = aA + bI = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}.$$

Note that in this simple case, the sign of β determines the direction of the rotation: $\beta > 0$ implies clockwise rotation and $\beta < 0$ implies anticlockwise rotation. More generally, to find the direction of rotation one can calculate Ax for a vector x, and use the fact that $x' = Ax$ is a vector tangent to the orbit through x at x.

We examine the behavior of the system for various values of α. Various trajectories are plotted in Figure 8.5.

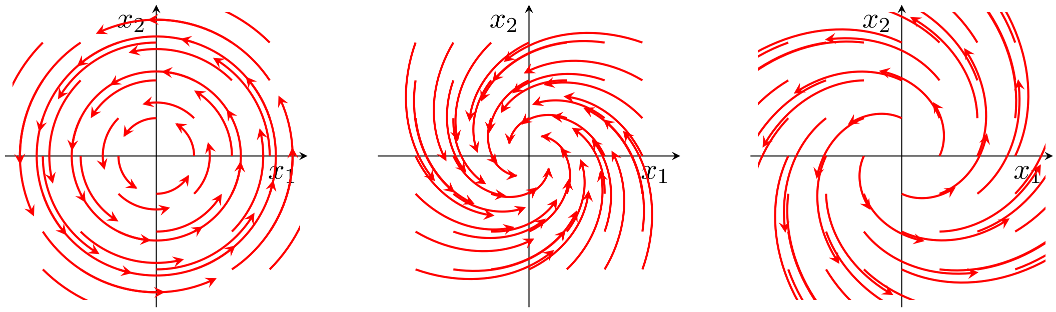
Case 1: $\alpha = 0$. The system simplifies to

$$x(t) = \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix} = \begin{pmatrix} x_0^{(1)} \cos\beta t - x_0^{(2)} \sin\beta t \\ x_0^{(2)} \cos\beta t + x_0^{(1)} \sin\beta t \end{pmatrix}.$$

In this case, $\|x(t)\| = \|x(0)\|$, so the distance from the origin remains constant for all t. The fixed point $x = 0$ is a stable fixed point but not asymptotically stable. Every $x \in \mathbb{R}^2 \setminus \{0\}$ is a periodic point with period $T = \frac{2\pi}{\beta}$.

Case 2: $\alpha < 0$. In this case, $\|x(t)\| = e^{\alpha t}\|x(0)\| \to 0$ as $t \to \infty$. Therefore, $x = 0$ is a stable and asymptotically stable fixed point.

Case 3: $\alpha > 0$. Similar to Case 2, $\|x(t)\| = e^{\alpha t}\|x(0)\| \to +\infty$ as $t \to \infty$. Therefore, $x = 0$ is an unstable and repelling fixed point. □



(a) $\alpha = 0$.   (b) $\alpha < 0$.   (c) $\alpha > 0$.

Figure 8.5: The trajectories of three cases from Example 8.26 based on the value of α.

Chapter 9

Non-linear ODEs

As we saw in Chapter 8, we can often use the solution of an ODE to define a flow Φ. If the ODE is given by

$$x' = f(x),$$

then we say the the autonomous function f *generates* the flow, denoted by Φ_f. Recall that $\bar{x}$ is a fixed point of a flow Φ if and only if $\Phi(t, \bar{x}) = \bar{x}$ for all $t \in \mathbb{R}$, and furthermore, $\Phi(t, \bar{x}) = \bar{x}$ if and only if $f(\bar{x}) = 0$.

Definition 9.1. Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^n$ be a C^1-function. A fixed point $\bar{x}$ of Φ_f is called **hyperbolic** if all eigenvalues of $Df(\bar{x})$ have nonzero real part.

Theorem 9.2. *Let $f : U \to \mathbb{R}^n$ be a C^1-function, and let $\bar{x}$ be a hyperbolic fixed point of the flow Φ_f.*

1. *The fixed point $\bar{x}$ is asymptotically stable if and only if each eigenvalue λ of $Df(\bar{x})$ satisfies $\mathrm{Re}(\lambda) < 0$.*

2. *The fixed point $\bar{x}$ is unstable and repelling if and only if each eigenvalue λ of $Df(\bar{x})$ satisfies $\mathrm{Re}(\lambda_i) > 0$.*

3. *The fixed point $\bar{x}$ is a saddle point if and only if there exist eigenvalues λ, λ' of $Df(\bar{x})$ with $\mathrm{Re}(\lambda) < 0$ and $\mathrm{Re}(\lambda') > 0$.*

The real value of Theorem 9.2 is that we understand the behavior of such systems *without* needing to find a solution.

Example 9.3. Consider the system

$$\begin{cases} x' = -2x - y^2, \\ y' = -x^2 - y. \end{cases}$$

In other words, the system is defined by the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, where $f(x, y) = (-2x - y^2, -x^2 - y)$. It is clear that $(x, y) = (0, 0)$ is a fixed point. The Jacobian at $(0, 0)$ is

$$Df(0,0) = \begin{pmatrix} -2 & -2y \\ -2x & -1 \end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, as the eigenvalues are both real and negative. From Theorem 9.2, the fixed point $(0, 0)$ is asymptotically stable. $\square$

Example 9.4 (van der Pol's equation)**.** For $\mu \in \mathbb{R}$, consider the differential equation

$$x'' - \mu(1 - x^2)x' + x = 0. \tag{9.1}$$

Setting $x_1 = x$ and $x_2 = x'$, we rewrite (9.1) as the system

$$\begin{cases} x_1' = x_2 \\ x_2' = \mu(1 - x_1^2)x_2 - x_1. \end{cases} \tag{9.2}$$

Like in Example 9.3, the system in (9.2) can be rewritten as $x' = f(x)$, where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x_1, x_2) = (x_2, \mu(1 - x_1^2)x_2 - x_1)$. Observe that $(0, 0)$ is, again, a fixed point. Thus,

$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ -2\mu x_1 x_2 - 1 & \mu(1 - x_1^2) \end{pmatrix}\Bigg|_{(x_1,x_2)=(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} =: A.$$

Now we need to determine the eigenvalues of A. The characteristic polynomial is

$$p_A(\lambda) = -\lambda(\mu - \lambda) + 1 = \lambda^2 - \mu\lambda + 1,$$

whose zeroes (and thus the eigenvalues) are

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

The eigenvalues are real if $|\mu| \geq 2$. In this case, using the fact that $\sqrt{\mu^2 - 4} < \mu$, we see that for $\mu \leq -2$, the eigenvalues are both negative, and for $\mu \geq 2$, the eigenvalues are both positive. On the other hand, if $|\mu| < 2$, then the eigenvalues are complex, with real part having the same sign as μ. In summary,

- if $\mu > 0$, then $\mathrm{Re}(\lambda) > 0$, so the origin is an unstable and repelling fixed point;
- if $\mu < 0$, then $\mathrm{Re}(\lambda) < 0$, so the origin is an asymptotically stable fixed point.

If the scalar $\mu = 0$, the system (9.2) becomes

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which is a linear system. The eigenvalues of the matrix are $\pm i$. Let $f(z) = e^{tz}$ and $q(z) = az + b$. Then the system

$$\begin{cases} f(i) &= q(i), \\ f(-i) &= q(-i), \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} ai + b &= e^{it}, \\ -ai + b &= e^{-it}, \end{cases}$$

has solution $a = \sin t$ and $b = \cos t$. The solution is then

$$e^{tA} = q(A) = aA + bI = \sin t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

so

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.$$

Each orbit is a rotation around the origin (with no scaling). Thus, in this case the origin is a stable (neither attracting nor repelling) fixed point. □

9.1 Lyapunov functions

Lyapunov functions are another tool for analyzing the stability of fixed points of non-linear dynamical systems. We will define these functions soon, but there is no general method for constructing Lyapunov functions. However, in many common scenarios, we can use these to determine stability.

Definition 9.5. Let $L : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function, and suppose $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the Lipschitz condition. The **orbital derivative** of L in the direction of f is the function $L'_f : \mathbb{R}^d \to \mathbb{R}$ given by

$$L'_f(x) := \sum_{k=1}^{d} \frac{\partial L}{\partial x_k}(x) f_k(x) = \text{grad } L(x) \cdot f(x).$$

Thus we can think of $L'_f(x)$ as the derivative of $L(x)$ in the tangent direction $f(x)$ for the orbit through x. This derivative is also known as the directional derivative of L in the direction of (the tangent of) f and be identified as the Lie derivative of L along f. Sometimes $L'_f(x)$ is also referred to as the derivative of L with respect to the system $x' = f(x)$. If $x(t)$ is a solution to $x' = f(x)$ then the chain rule gives

$$\frac{d}{dt} L(x(t)) = \sum_{k=1}^{d} \frac{\partial L}{\partial x_k}(x(t)) x'_k(t) = L'_f(x(t)).$$

Definition 9.6. Let x_0 be a fixed point of f, and let $\Omega \subseteq \mathbb{R}^d$ be an open neighborhood containing x_0. A continuously differentiable function $L : \Omega \to \mathbb{R}$ is called a **Lyapunov function** for the system $x' = f(x)$ if $L(x_0) = 0$ and

(i) for all $x \in \Omega \setminus \{x_0\}$, $L(x) > 0$ and

(ii) for all $x \in \Omega \setminus \{x_0\}$, $L'_f(x) \leq 0$.

We call L a **strict Lyapunov function** if (ii) is always a strict inequality.

Example 9.7. We consider Lyapunov function for the system given by

$$\begin{aligned} x' &= -2x - y \\ y' &= -x^2 - 4x - y. \end{aligned}$$

Let $f(x, y) := (-2x - y, -x^2 - 4x - y)$. The fixed points of the system are (x, y) such that $f(x, y) = (0, 0)$. Thus, fixed points satisfy

$$y = -2x, \qquad\qquad y = -x^2 - 4x.$$

Since y is determined by x, it follows that the x-value of all fixed points satisfy

$$0 = x^2 + 2x = x(x + 2).$$

Therefore, there are two fixed points of the system: $(0, 0)$ and $(-2, 4)$.

We define a Lyapunov function for $x_0 = (-2, 4)$. Set

$$L(x, y) = (x + 2)^2 + (y - 4)^2,$$

which satisfies $L(x_0) = 0$ and $L(x) > 0$ for $x \in \mathbb{R}^2 \setminus \{x_0\}$. To determine if the last condition is satisfied, we take the orbital derivative along f:

$$\begin{aligned} L'_f(x, y) &= (2x + 4, 2y - 8) \cdot (-2x - y, -x^2 - 4x - y) \\ &= -(2x + 4)(2x + y) - (2y - 8)(x^2 + 4x + y) \\ &= -2x^2 y + 4x^2 - 10xy + 24x - 2y^2 + 4y. \end{aligned}$$

The gradient of L'_f is $(-4x(y-2)-10y+24, -2(x^2+5x+2y-2))$. The function $L'_f(x,y)$ has a local maximum at $(-2,4)$, so there exists an open neighborhood Ω around $(-2,4)$ such that L is a Lyapunov function. $\square$

The question left hanging now is: *how does this help us analyze dynamical systems*? Right now, it just seems like these are mysterious functions. The next proposition proves that the existence of a Lyapunov function implies stability.

Proposition 9.8. *If the system $x' = f(x)$ has a Lyapunov function in a neighborhood Ω of the fixed point x_0, then x_0 is a stable fixed point.*

Proof. Let $\varepsilon > 0$. Without loss of generality, we can assume that $B_\varepsilon(x_0) \subseteq \Omega$. We need to show that there exists $\delta > 0$ such that $\|x(t) - x_0\| < \varepsilon$ for all t whenever $\|x(0) - x_0\| < \delta$. Let

$$S_\varepsilon = \{x \in \Omega : \varepsilon/2 \leq \|x - x_0\| \leq \varepsilon\} = \overline{B_\varepsilon(x_0)} \setminus B_{\varepsilon/2}(x_0).$$

The set S_ε is closed and bounded,

$$\mu := \min_{x \in S_\varepsilon}\{L(x)\}$$

is well-defined and $\mu > 0 = L(x_0)$. As L is continuous, we may choose $\delta > 0$ with $\delta < \varepsilon/2$ such that $L(x) < \mu$ for all $\|x - x_0\| < \delta$.

Let $x(t)$ be a solution for the system $x' = f(x)$. The condition that $L'_f(x) \leq 0$ implies that $L(x)$ decreases along the orbits $x(t)$. Thus, $\|x(0) - x_0\| < \delta$ implies that $L(x(0)) < \mu$; hence, $L(x(t)) < \mu$ for all $t \geq 0$ because $L'_f(x) \leq 0$. Therefore, $\|x(t) - x_0\| < \varepsilon/2$ for all $t \geq 0$, and thus the fixed point $x = 0$ is stable. $\square$

The next proposition takes Proposition 9.8 further to give conditions on asymptotically stable fixed points.

Proposition 9.9. *If the system $x' = f(x)$ has a strict Lyapunov function in a neighborhood Ω of the fixed point x_0, then x_0 is an asymptotically stable fixed point.*

Proof. Let $x(t)$ be a solution of $x' = f(x)$ and $L(x)$ a strict Lyapunov function. Let $\varepsilon > 0$ and take δ as in the proof of Proposition 9.8, so that for $\|x(0) - x_0\| \leq \delta$, $\|x(t) - x_0\| \leq \varepsilon/2$ for all $t \geq 0$. Since $L(x(t))$ is decreasing and bounded below, the limit

$$\lim_{t \to \infty} L(x(t))$$

exists; call it L_0. By the nonnegativity of L, it follows that $L_0 \geq 0$.

We will show that $L_0 = 0$, so suppose instead then that $L_0 > 0$. By the continuity of L, there exists $r > 0$ such that $r < \varepsilon/2$ and $L(x) = \|L(x)\| < L_0$ whenever $\|x - x_0\| < r$. Let

$$\Delta := \{x \in \mathbb{R}^d : r \leq \|x - x_0\| \leq \varepsilon/2\}.$$

As the set Δ is closed and bounded and L'_f is continuous, the number

$$k := \max_{x \in \Delta}\{L'_f(x)\}$$

exists. Since L is a strict Lyapunov function, $k \neq 0$, so $k < 0$. For $\|x(0) - x_0\| \leq \delta$, we have $x(t) \in \Delta$ for all $t \geq 0$. Thus $L'_f(x(t)) \leq k < 0$ for all $t \geq 0$. This implies that

$$\lim_{t \to \infty} L(x(t)) = -\infty,$$

which is a contradiction. Thus, $0 = L_0 = \lim_{t \to \infty} L(x(t))$, so by continuity of L, $\lim_{t \to \infty} x(t) = x_0$. This implies that x_0 is asymptotically stable. $\square$

Remark 9.10. One could write Propositions 9.8 and 9.9 only for the fixed point $x = 0$. Such statements are equivalent to the ones above as we may translate the system and then apply the Propositions. That is, we set $g(x) = f(x + \bar{x})$ so that $g(0) = f(\bar{x}) = 0$.

Example 9.11. We revisit Example 9.7. Recall the system there is given by

$$\begin{aligned} x' &= -2x - y \\ y' &= -x^2 - 4x - y, \end{aligned}$$

where $f(x, y) := (-2x - y, -x^2 - 4x - y)$. The fixed points are $(0, 0)$ and $(-2, 4)$.

We constructed a Lyapunov function for the fixed point $(-2, 4)$, namely:

$$L(x, y) = (x + 2)^2 + (x - 4)^2.$$

In fact, this is a strict Lyapunov function. Thus, by Proposition 9.9, the fixed point $(-2, 4)$ is asymptotically stable.

Note that it is not so easy to find a Lyapunov function for the fixed point $(0, 0)$. The Jacobian of f at $(0, 0)$ is

$$Df(0, 0) = \begin{pmatrix} -2 & -1 \\ -2x - 4 & -1 \end{pmatrix}\bigg|_{(x,y)=(0,0)} = \begin{pmatrix} -2 & -1 \\ -4 & -1 \end{pmatrix}.$$

The matrix $Df(0, 0)$ has one positive and one negative eigenvalue. Therefore, by Proposition 9.8, there *cannot* exist a Lyapunov function for the fixed point $(0, 0)$. □

Example 9.12. Consider the system

$$\begin{aligned} x' &= -x^3 + y^2 \\ y' &= -2xy - y \end{aligned}$$

This time, the origin is the only fixed point. Writing the system, as usual, as $x' = f(x)$, we calculate the Jacobian at $(0, 0)$ to be

$$Df(0, 0) = \begin{pmatrix} -3x^2 & -2y \\ -2y & -2x - 1 \end{pmatrix}\bigg|_{(x,y)=(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us try to find a Lyapunov function; we look for something of the form $L(x, y) = Ax^2 + By^2$, with $A, B > 0$. Note that this already satisfies the first condition of Lyapunov functions. Then

$$\begin{aligned} L'_f(x, y) &= 2Ax(-x^3 + y^2) + 2By(-2xy - y) \\ &= -2Ax^4 + (2A - 4B)xy^2 - 2By^2. \end{aligned}$$

If we can get $2A - 4B = 0$, then we can satisfy the second condition of Lyapunov functions. Setting $A = 2$ and $B = 1$ yields

$$L'_f(x, y) = -4x^4 - 2y^2,$$

which is negative for for all $(x, y) \neq (0, 0)$. Thus $L(x, y) = 2x^2 + y^2$ is a strict Lyapunov function for our system, and $(0, 0)$ is an asymptotically stable fixed point of the system. □

Finding a Lyapunov function for a system can be a process of trial and error, and "all the functions I've tried are not Lyapunov functions" is not a proof that a given fixed point is not stable! It actually suggests that you might try a different approach entirely: show the fixed point is *not* stable. We can try forming the Jacobian and calculating eigenvalues, as mentioned above, or use the following alternative method to verify instability.

Proposition 9.13. *Let $x' = f(x)$ be a system with fixed point x_0, and suppose Ω is an open neighborhood about x_0. Let Ω' be an open subset of Ω and $G : \Omega \to \mathbb{R}$ a continuously differentiable function such that*

(i) $x_0 \in \overline{\Omega'}$;

(ii) $G(x) > 0$ *and* $G'_f(x) > 0$ *for all* $x \in \Omega'$;

(iii) $G(x) = 0$ *for all* $x \in \partial\Omega' \cap \Omega$.

Then x_0 is an unstable fixed point.

Proof. By definition, x_0 is an unstable fixed point of the system if there is an open ball around x_0 with the property that there are orbits starting arbitrarily close to x_0 that leave the ball. Choose $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(x_0)} = \{x \in \mathbb{R}^d : \|x - x_0\| \leq \varepsilon\} \subseteq \Omega.$$

Since $x_0 \in \overline{\Omega'}$, we are done if we can show that each orbit starting in Ω' leaves $B_\varepsilon(x_0)$. To this end, let $x(t)$ be a solution to $x' = f(x)$ with $x(0) \in \Omega'$. As $G(x(0)) > 0$ we can choose $\delta > 0$ such that $G(x(0)) > \delta$, and then define

$$\Delta = \left\{x \in \Omega' \cap \overline{B_\varepsilon(x_0)} : G(x) \geq \delta\right\}.$$

The set Δ is closed and bounded, so that the number

$$k = \min_{x \in \Delta}\{G'_f(x)\} > 0$$

exists. Suppose that $x(t) \in \Delta$ for all $t \geq 0$. Then by compactness of Δ,

$$x_* := \lim_{t\to\infty} x(t) \in \Delta.$$

But this means that $k \leq G'_f(x(t))$ for all t, which implies that

$$G\left(\lim_{t\to\infty} x(t)\right) = \lim_{t\to\infty} G(x(t)) = \infty.$$

This is a contradiction, so there exists $t > 0$ such that $x(t) \notin \Delta$. Since $G(x(t)) > G(x(0)) \geq \delta$, this implies that $x(t) \notin B_\varepsilon(x_0)$. □

Example 9.14. Consider the system

$$\begin{aligned} x' &= x + xy \\ y' &= -2y + xy. \end{aligned}$$

The origin is a fixed point of the system. Let $G(x, y) = x^2 - y^2$. Then $G(x, y) = 0$ if and only if $x^2 = y^2$, which is equivalent to $|x| = |y|$. We are looking for Ω and Ω' to satisfy the conditions of Proposition 9.13. First, we set

$$U = \left\{(x, y) \in \mathbb{R}^2 : x > |y|\right\}.$$

For all $(x, y) \in U$, $G(x, y) > 0$ and for all $(x, y) \in \partial U$, $G(x, y) = 0$. The orbital derivative of G along f is

$$\begin{aligned} G'_f(x, y) &= 2x(x + xy) - 2y(-2y + xy) \\ &= 2x^2 + 2x^2y + 4y^2 - 2xy^2 \\ &= 2x^2 + 4y^2 + 2x^2y - 2xy^2. \end{aligned}$$

We need that $G'_f(x, y) > 0$ for all $(x, y) \in \Omega'$. The only (possibly) negative parts of the above expression are the last two terms, so we look for the minimum of the function $F(x, y) := 2x^2y - 2xy^2$ on U. The function F has a critical point only $(0, 0)$, and $F(0, 0) = 0$. We check the boundaries:

(i) if $y = x$, then $F(x, x) = 2x^3 - 2x^3 = 0$, and

(ii) if $y = -x$, then $F(x, -x) = -2x^3 - 2x^3 = -4x^3$.

So we need to bound our region. Let $\Omega = B_1(0)$, and set

$$\Omega' = U \cap \Omega = \{(x, y) \in \mathbb{R}^2 : |y| < x < 1\}.$$

Then the conditions of Proposition 9.13 are fulfilled, and thus the origin is an unstable fixed point of the system. $\square$

Example 9.15. Consider the system

$$\begin{aligned} x' &= x^2 + 2x^2 y, \\ y' &= xy + x^3, \end{aligned}$$

and let $G(x, y) = Ax^2 - By^2$. Again, the origin is a fixed point of the system. The orbital derivative of G along f is

$$\begin{aligned} G'_f(x, y) &= 2Ax(x^2 + 2x^2 y) - 2By(xy + x^3) \\ &= 2Ax^3 + 4Ax^3 y - 2Bxy^2 - 2Bx^3 y \\ &= 2Ax^3 + x^3 y(4A - 2B) - 2Bxy^2 \end{aligned}$$

Setting $A = 1$ and $B = 2$, $G(x, y) = x^2 - 2y^2$ and

$$G'_f(x, y) = 2x^3 - 4xy^2 = 2x(x^2 - 2y^2).$$

For $\Omega = B_1(0)$, let $\Omega' = \{x \in \Omega : x^2 > 2y^2, x > 0\}$, which is an open set of Ω. Then the conditions of Proposition 9.13 are satisfied. Therefore, the origin is an unstable fixed point. $\square$

Chapter 10

Dynamic Programming

A discrete dynamic optimization problem can be modeled mathematically as follows. Let $\mathbb{G} = \{t_0, \dots, t_N\} \subseteq \mathbb{R}$ be a *lattice* of $N+1$ fixed points. The **state trajectory function** is denoted by $y : \mathbb{G} \to \mathbb{R}^n$, and the **control function** is denoted by $u : \mathbb{G} \to \mathbb{R}^m$. With $\varphi : \mathbb{G} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, the **objective function** is

$$f(y, u) := \sum_{j=0}^{N} \varphi(t_j, y(t_j), u(t_j)).$$

For $\psi : \mathbb{G} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the **dynamic equations** are, for $j \in \{0, \dots, N-1\}$,

$$y(t_{j+1}) = \psi(t_j, y(t_j), u(t_j)).$$

We require that the state trajectory function always map into the (nonempty) **state restriction sets** $Y(t_j) \subseteq \mathbb{R}^n$. That is, for all $t_j \in \mathbb{G}$, $y(t_j) \in Y(t_j)$. Furthermore, we require that the control functions always map into the (nonempty) **control restriction sets** $U(t_j, y(t_j)) \subseteq \mathbb{R}^m$, so for all $t_j \in \mathbb{G}$, $u(t_j) \in U(t_j, y(t_j))$.

The discrete optimization problem is the following.

Problem 10.1 (Discrete optimization problem)**.** Given a state trajectory function $y : \mathbb{G} \to \mathbb{R}^n$ and a control function $u : \mathbb{G} \to \mathbb{R}^m$, minimize the sum

$$\sum_{j=0}^{N} \varphi(t_j, y(t_j), u(t_j))$$

subject to the conditions:

$$\begin{aligned}
&(\forall j \in \{0, \dots, N-1\}) && y(t_{j+1}) = \psi(t_j, y(t_j), u(t_j)),\\
&(\forall j \in \{0, \dots, N\}) && y(t_j) \in Y(t_j),\\
&(\forall j \in \{0, \dots, N\}) && u(t_j) \in U(t_j, y(t_j)).
\end{aligned}$$

Usually, the sets $Y(t_j)$ and $U(t_j, y(t_j))$ are implicitly given as solution sets to the equations determined by ψ.

Example 10.2 (Warehousing)**.** A firm wants to store a product for a fixed number of times $t_0 < \cdots < t_N$ with minimal cost. Denote by $u_j \geq 0$ the delivered quantity at t_j, by $r_j \geq 0$ the demands in the interval $[t_j, t_{j+1})$, and y_j the quantity of goods stored at t_j (directly before delivery). Therefore, for $j \in \{0, \dots, N-1\}$,

$$y_{j+1} = y_j + u_j - r_j.$$

We also require that the demands also be met, so $y_{j+1} \geq 0$ for $j \in \{0, \ldots, N-1\}$. Without loss of generality, we assume that $y_0 = y_N = 0$. The delivery costs at time t_j are modeled by

$$B(u_j) = \begin{cases} cu_j + K & \text{if } u_j > 0, \\ 0 & \text{if } u_j = 0, \end{cases}$$

where c is the cost per unit of product and K is the fixed cost for delivery. For $x \in \mathbb{R}^\ell$, let [1]

$$\delta(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Therefore, we can write more compactly $B(u_j) = cu_j + K\delta(u_j)$. The warehousing costs fall at the end of the each time interval and are given by hy_{j+1}. Therefore the total cost is

$$\sum_{j=0}^{N-1} cu_j + K\delta(u_j) + hy_{j+1}. \tag{10.1}$$

We want to write the total cost in (10.1) with just j (and no $j+1$). To that end, notice that since $y_0 = Y_N = 0$,

$$\sum_{j=0}^{N-1} y_{j+1} = \sum_{j=1}^{N} y_j = \sum_{j=0}^{N-1} y_j.$$

Therefore, we can state the warehousing optimization problem as follows.

Problem 10.3. Minimize

$$\sum_{j=0}^{N-1} K\delta(u_j) + hy_j$$

subject to the constraints

$$\begin{aligned} &(\forall j \in \{0, \ldots, N-1\}) && y_{j+1} = y_j + u_j - r_j, \\ & && y_0 = y_N = 0, \\ &(\forall j \in \{1, \ldots, N\}) && y_j \geq 0, \\ &(\forall j \in \{0, \ldots, N-1\}) && u_j \geq 0. \end{aligned}$$

Example 10.4 (The Knapsack Problem)**.** This simple problem is found in various applications. Each u_j describes an object one wants to put in their knapsack. Each object is given a weight w_j and a value v_j. The knapsack has a total allowed weight W, and the goal is increase the value without going beyond the weight. In other words, maximize

$$\sum_{j=1}^{N} v_j u_j$$

subject to the constraints that $u_j \in \{0, 1\}$ and

$$\sum_{j=1}^{N} w_j u_j \leq W.$$

However, we can state this as a dynamical optimization problem. We let y_j denote the remaining weight available in the knapsack after time t_j, and let y_{N+1} denote the remaining weight at the end.

[1]This is called the Kronecker delta function.

Problem 10.5. Maximize

$$\sum_{j=1}^{N} v_j u_j$$

subject to

$$\begin{aligned}
(\forall j \in \{1,\dots,N\}) \qquad & y_{j+1} = y_j - w_j u_j, \\
& y_1 = W, \\
(\forall j \in \{1,\dots,N\}) \qquad & y_j \geq 0, \\
(\forall j \in \{1,\dots,N\}) \qquad & u_j \begin{cases} \in \{0,1\} & \text{if } y_j \geq w_j, \\ = 0 & \text{if } y_j < w_j. \end{cases}
\end{aligned}$$

10.1 The Principle of Optimality

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