Matrix exponentials: basics

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$x' = Ax$. $x \in \mathbb{R}^d$.

In $d=1$: $x' = Ax$, where $A = (a)$

$$x(t) = Ce^{at}$$

For $d \geq 1$, the solution is

$$x(t) = x_0 e^{tA}$$

How do we exponentiate matrices? e^A?

Recall,

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\boxed{e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}}$$

$x' = Ax$

$A \in Mat_{d \times d}(\mathbb{R})$

Notice if A is not square, this will not work.

<u>Convergence:</u> The power series for e^x converges for all $x \in \mathbb{R}$. For all

e^x converges for all $x \in \mathbb{R}$. For all $A \in Mat_{d\times d}(\mathbb{R})$, e^A converges.

For $A, B \in Mat_{d\times d}(\mathbb{R})$, $\|AB\| \le \|A\|\,\|B\|$.

For all $n \in \mathbb{N}$,

$$\|A^n\| \le \|A\|^n.$$

If $(e^A)_{ij}$ is the (i,j)-entry, then

$$|(e^A)_{ij}| \le \sum_{n=0}^{\infty} \left| \frac{(A^n)_{ij}}{n!} \right| \le \sum_{n=0}^{\infty} \frac{\|A^n\|}{n!}$$

$$\le \boxed{\sum_{n=0}^{\infty} \frac{\|A\|^n}{n!}}$$

Converges for all $\mathbb{R}$.

Prop 8.4: For $A \in Mat_{d\times d}(\mathbb{R})$

$$\frac{d}{dt} e^{tA} = A \cdot e^{tA}.$$

(Recall, $x' = Ax$ $\quad \frac{dx}{dt} = A x$)

proof:

$$\frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(tA)^m}{m!}$$

$\boxed{m! = 1\cdot 2\cdot 3 \cdots m}$

$$= \sum_{m=1}^{\infty} \frac{\boxed{m}\, t^{m-1} A^m}{m!}$$

$$= \sum_{m=1} \frac{\boxed{m} \cdots A}{m!}$$

$$= A \cdot \sum_{m=1}^{\infty} \frac{t^{m-1} A^{m-1}}{(m-1)!}$$

$$= A \cdot \sum_{m=0}^{\infty} \frac{t^m A^m}{m!}$$

$$= A \cdot e^{tA}.$$

Ex. 8.2: If A is diagonal, then we can compute e^A directly:

$$A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{dd} \end{pmatrix}.$$

$$\boxed{A^m = \begin{pmatrix} a_{11}^m & & & 0 \\ & a_{22}^m & & \\ & & \ddots & \\ 0 & & & a_{dd}^m \end{pmatrix}}$$

$$e^A = \sum^{\infty} \frac{A^m}{m!}$$

$$e^{A} = \sum_{m=0} \frac{A}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \underline{\begin{pmatrix} a_{11}^{m} & & 0 \\ & a_{22}^{m} & \\ & & \ddots & \\ 0 & & & a_{dd}^{m} \end{pmatrix}}$$

$$= \begin{pmatrix} \underline{\sum_{m=0}^{\infty} \frac{a_{11}^{m}}{m!}} & & & 0 \\ & \underline{\sum_{m=0}^{\infty} \frac{a_{22}^{m}}{m!}} & & \\ & & \ddots & \\ 0 & & & \underline{\sum_{m=0}^{\infty} \frac{a_{dd}^{m}}{m!}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{a_{11}} & & & 0 \\ & e^{a_{22}} & & \\ & & \ddots & \\ 0 & & & e^{a_{dd}} \end{pmatrix}.$$

Suppose $A = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \pi & \\ 0 & & & 2 \end{pmatrix}$.

$$e^{A} = \begin{pmatrix} e & & & 0 \\ & e^{-1} & & \\ & & e^{\pi} & \\ 0 & & & e^{2} \end{pmatrix}.$$