Implications of Lyapunov functions

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$L : \mathbb{R}^d \to \mathbb{R} \quad (f : \mathbb{R}^d \to \mathbb{R}^d)$

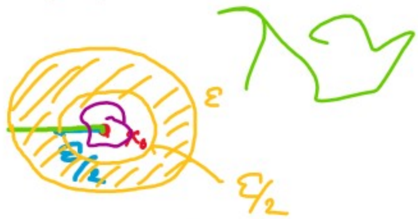
1. $L(x_0) = 0$,
2. $\forall x \in \Omega \setminus \{x_0\}$, $L(x) > 0$,
3. $\forall x \in \Omega \setminus \{x_0\}$, $L_f'(x) \leq 0$.

(strict means 3 is strict.)

Prop 8.9 If the system $x' = f(x)$ has a L.f. in a neighborhood Ω of a fixed x_0, then x_0 is stable.

proof: Let $\varepsilon > 0$. Without loss of generality $B_\varepsilon(x_0) \subseteq \Omega$. We show $\exists \delta > 0$ s.t.

$\|x(t) - x_0\| < \varepsilon$ for all t whenever $\|x(0) - x_0\| < \delta$.



Let

$$S_\varepsilon = \{x \in \Omega : \tfrac{\varepsilon}{2} \leq \|x - x_0\| \leq \varepsilon\}$$
$$= \overline{B_\varepsilon(x_0)} \setminus B_{\varepsilon/2}(x_0).$$

The set S_ε is closed and bounded. Then,

$$\mu = \min_{x \in S_\varepsilon} \{L(x)\}$$

is well-defined. We know that

$$\mu > 0 = L(x_0).$$

Since L is continuous, we may choose $\delta > 0$ with $\boxed{\delta < \varepsilon/2}$ s.t

$$L(x) < \mu \qquad \|L(x) - L(x_0)\| < \varepsilon$$

for all $\|x - x_0\| < \delta$. $\qquad L : \mathbb{R}^d \to \mathbb{R}$

Let $x(t)$ be a solution. The condition

Let $x(t)$ be a solution. The condition $L_f'(x) \leq 0$ implies that $L(x)$ decreases alongs the orbit $x(t)$. Thus, $\|x(0) - x_0\| < \delta$ implies that $L(x(0)) < \mu$. Hence,

$$L(x(t)) < \mu$$

for all $t \geq 0$. Thus, $x(t)$ is not in S_ε for all $t \geq 0$. Therefore, $\|x(t) - x_0\| < \boxed{\varepsilon/2}$ for all $t \geq 0$, and thus the fixed is stable. ▨

$\|x(t) - x_0\| < \delta < \varepsilon/2$