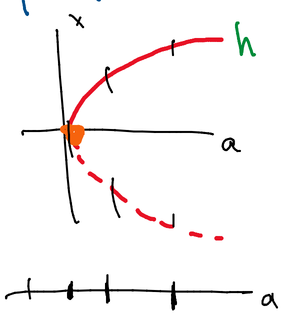
Lecture 9

Friday, May 22, 2020 8:34 AM

Saddle-node bifurcations:

Two fixeds points "collide" and disappear.

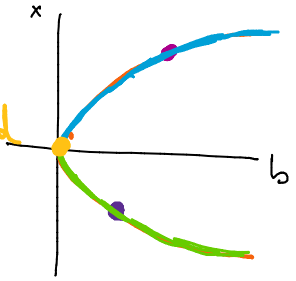


h is obtained by IFT. Unfortunately the IFT says very little about where this h is defined.

$h(b) = \sqrt{b}$

No open interval

$h(b) = -\sqrt{b}$



$f_b(x) = -x^2 + x + b$

$\boxed{x^2 - b = 0}$

fixed points must satisfy

The problem here is that $x^2 - b = 0$ is not a function $x = g(b)$.

If we want to express the orange curve as a function of b near the purple point, then

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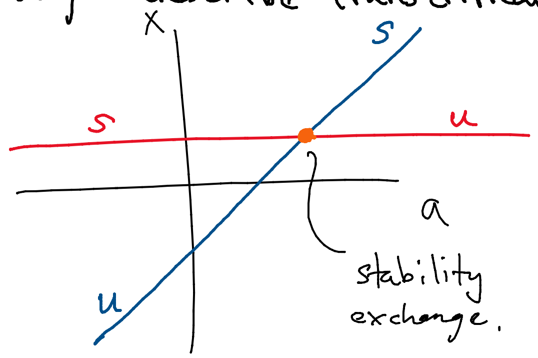
Q: How do you describe transcritical?

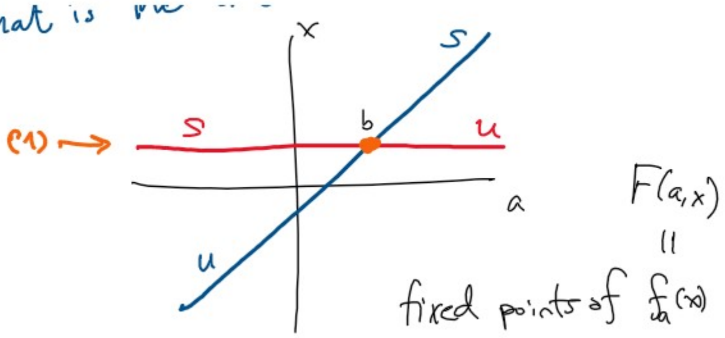


There is a stability exchange between two fixed points.

Q: What do you have that may have transcritical?

Q: What is Theorem 4.6 about?

Q: What is the ...



$F(a,x)$
$\parallel$
fixed points of $f_a(x)$

→ 1. $\forall a \in J$, $\underline{F(a,\bar{x}) = \bar{x}}$

$\bar{x}$ is a fixed pt for all J.

→ 2. $\frac{\partial F}{\partial x}(\bar{a},\bar{x}) = 1$ (not a hyperbolic fixed point)

Could be a bifurcation point.

→ 3. $\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x}) \neq 0$,

Tells us how deriv. is changing.

4. $\frac{\partial}{\partial a}\left(\frac{\partial F}{\partial x}(a,x)\right)\Big|_{\substack{a=\bar{a}\\ x=\bar{x}}} \neq 0.$

This is enabling the use of the IFT.

Comparing (4) of Theorem 4.6 with (4) of Theorem 4.4, these both serve the same purpose: enabling the use of IFT.

In Thm 4.4, $G = F(a,x) - x$; we applied IFT on G just fine.
Here (in Thm 4.6), this does not work. We need to check: either

$$\underbrace{\frac{\partial G}{\partial x}(\bar{a},\bar{x}) \neq 0}_{\text{because of (2) this is } 0} \quad \text{or} \quad \underbrace{\frac{\partial G}{\partial a}(\bar{a},\bar{x}) \neq 0.}_{\Rightarrow \frac{\partial F}{\partial a}(\bar{a},\bar{x}) = 0.}$$

By (1) $\frac{\partial F}{\partial a}(\bar{a},\bar{x}) = 0.$

$$\boxed{H(a,x) = \begin{cases} \dfrac{F(a,x) - x}{x - \bar{x}} & x \neq \bar{x}, \\[2ex] \dfrac{\partial F}{\partial x}(a,\bar{x}) - 1 & x = \bar{x}. \end{cases}}$$

Because G doesn't work (IFT ...

~~Because G doesn't work~~ (IFT does not apply), we define a better one.

Thm 4.4: Saddlenode bifurcation.

Outline of proof

1. Define G so that IFT applies
2. Then using chain rule, calc. h' and h''.

Thm 4.6: Transcritical bifurcation

Outline of proof

1. Define H so that IFT applies. (This is the majority of the proof)
2. Then using chain rule, calc. g'.

Examples: 4.1 (Saddle-node) $f_b(x) = -x^2 + x + b$

4.5 (Transcritical) $f_a(x) = a\,x(1-x)$.

Saddlenode bifurcation analysis.

Recall $h(x)$ is the function from IFT that expresses $G(a,x)$ as a real-valued function.

$$h'(\bar{x}) = 0$$

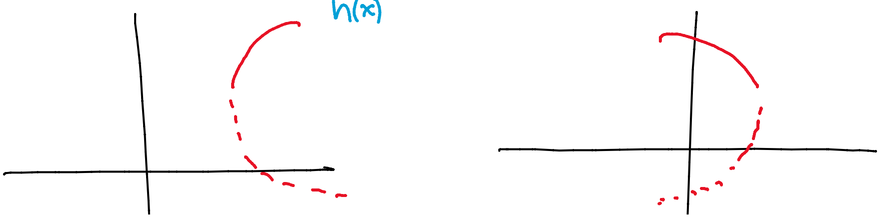
$$h''(\bar{x}) = -\frac{\frac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x})}{\frac{\partial F}{\partial a}(\bar{a},\bar{x})}.$$

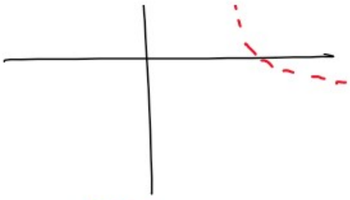
Close by $\bar{x}$ analysis

assumption

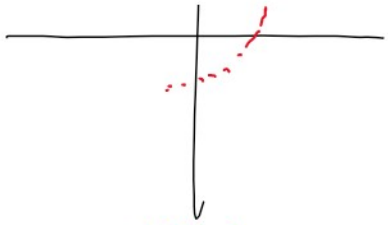
Case 1. : $f_{\bar{a}}''(\bar{x}) < 0$. If $\bar{x} > x$, $\boxed{f_{\bar{a}}'(\bar{x}) = 1}$ this implies that $f_{h(x)}'(x) > 1$. If $\bar{x} < x$, this implies that

$$f_{h(x)}'(x) < 1.$$



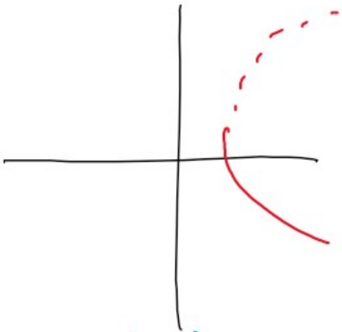


$h''(\bar{x}) > 0$

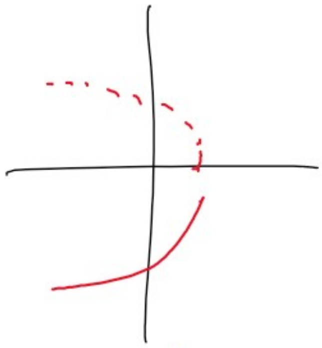


$h''(\bar{x}) < 0$

Use the same analysis for $f''_{\bar{a}}(\bar{x}) > 0$.



$h''(\bar{x}) > 0$



$h''(x) > 0$

Transcritical analysis.

Recall bifurcation at $f_{\bar{a}}(\bar{x})$, and

$$f'_{\bar{a}}(\bar{x}) = 1.$$

Case 1: $x = \bar{x}$. Because of assumptions

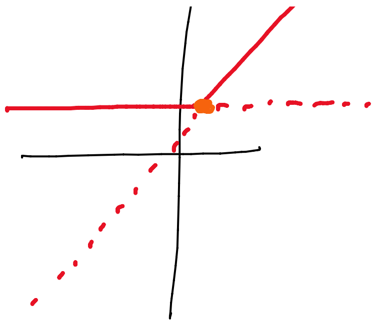
$$\frac{\partial F}{\partial x}(\bar{a}, \bar{x}) = 1 \qquad \frac{\partial^2 F}{\partial a \partial x}(\bar{a}, \bar{x}) \neq 0.$$

Thus, $\partial F / \partial x$ is changing with respect to a, but by Thm 3.5, the stability must change right at $a = \bar{a}$. In other words, we have to have an exchange of stability.

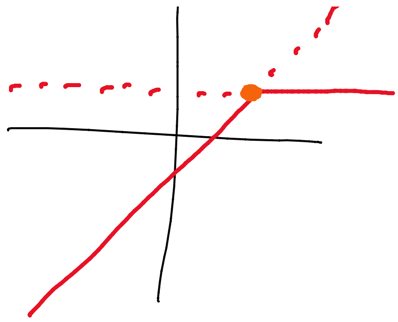
Case 2. $x = g(a)$. Use a Taylor expansion to show: for a near $\bar{a}$:

$$\frac{\partial F}{\partial x}(a, g(a)) \approx 1 - \boxed{\frac{\partial^2 F}{\partial x \partial a}(\bar{a}, \bar{x})}(a - \bar{a}).$$

Again, we see the stability change, but in the opposite order.



$$\frac{\partial^2 F}{\partial x \partial a}(\bar{a},\bar{x}) > 0$$



$$\frac{\partial^2 F}{\partial x \partial a}(\bar{a},\bar{x}) < 0.$$