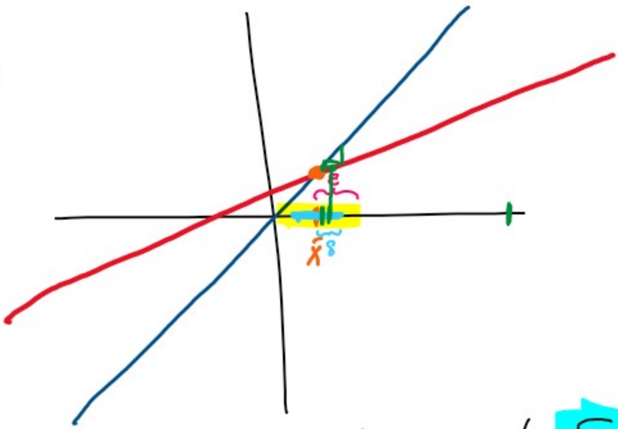
Lecture 8

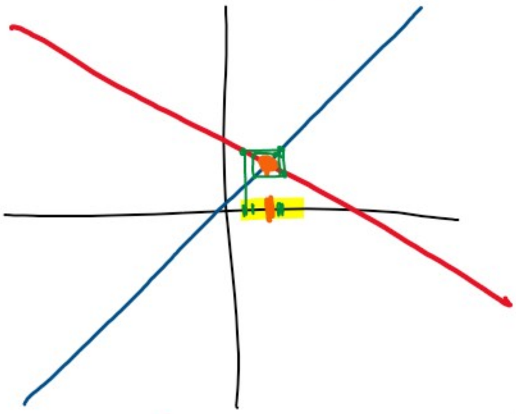
Tuesday, 19 May, 2020 14:05

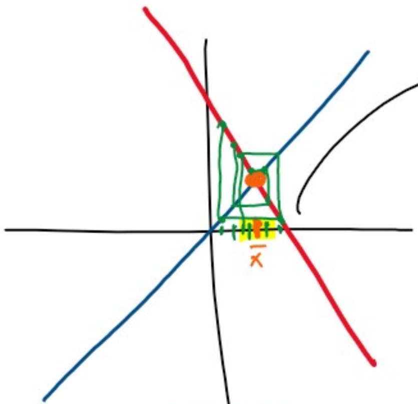
Stable:



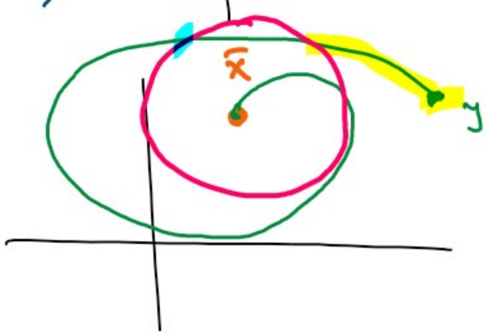
For every $\varepsilon>0$, there exists $\delta>0$
for all y, with $\|x-y\|<\delta$,
then $\boxed{\|f^n(y)-x\|<\varepsilon.}$

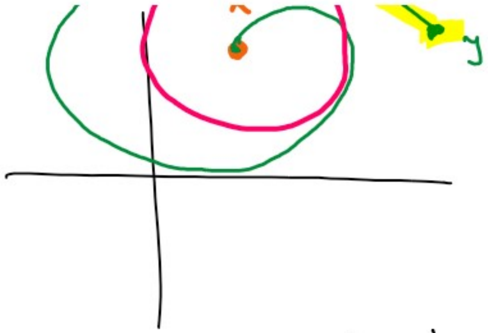
For every $\varepsilon>0$, we can find points close enough to x so that all iterations are within ε of x.





There is no δ such that all points within δ-distance from $\bar{x}$ stay within ε-distance from $\bar{x}$.

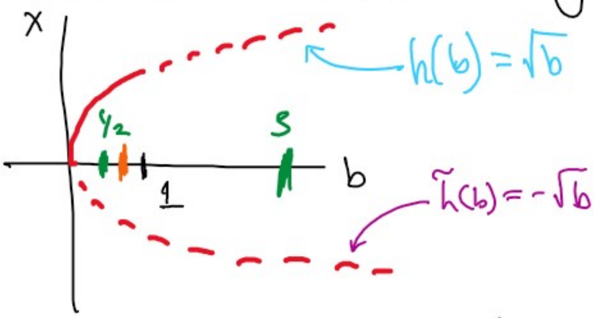




The point $\bar{x}$ is attracting if $\exists\, \delta > 0$ such that for all y with $\|\bar{x} - y\| < \delta$

$$\lim_{n \to \infty} f^n(y) = \bar{x}.$$

Last time we saw the bifurcation diagram

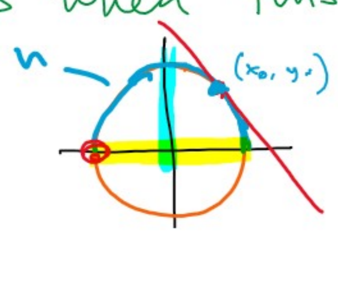


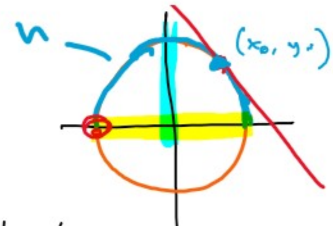
These diagrams depict the fixed points of a family $f_a(x)$, and usually solid lines indicate stable fixed points while dashed lines indicate unstable fixed points.

A major ingredient to understanding bifurcations is the Implicit Function Theorem. Basically if we have an equation

$$G(x, y) = 0 \qquad (x^2 + y^2 - 1 = 0)$$

and we want a "solution" function of the form $y = f(x)$, then the IFT tells when this is possible.





Thm (IFT) Let $G: \mathbb{R}^2 \to \mathbb{R}$ be cont. diff. on an open set $U \subseteq \mathbb{R}^2$. If for $(x_0, y_0) \in U$, $G(x_0, y_0) = 0$ and

$$\frac{\partial G}{\partial y}(x_0, y_0) \neq 0,$$

then there exists open intervals $I, J \subseteq \mathbb{R}$ with $x_0 \in I$, $y_0 \in J$, $I \times J \subseteq U$, and a cont. diff. function $h: I \to J$ such that $h(x_0) = y_0$ and for all $x \in I$

$$G(x, h(x)) = 0.$$

Applying chain rule to h:

$$h'(x_0) = \frac{-\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}.$$

The IFT gives us a way to locally express fixed points of $f_a(x)$ in terms of a.

Cor 4.3: Let $I, J \subseteq \mathbb{R}$ and $F: J \times I \to I$ be cont. diff. If for some $\bar{a} \in J$, $\bar{x} \in I$ $F(\bar{a}, \bar{x}) = \bar{x}$ and

$$\frac{\partial F}{\partial x}(\bar{a}, \bar{x}) \neq 1,$$

then there exists open intervals $K \subseteq I$ and $L \subseteq J$ with $\bar{a} \in K$, $\bar{x} \in L$ and a unique function $h: K \to L$ s.t. $h(\bar{a}) = \bar{x}$ and for all $a \in K$,

$$F(a, h(a)) = h(a).$$

$F(a,x) = f_a(x)$.

$f_{\bar{a}}(\bar{x}) = \bar{x}$ so $\bar{x}$ is a fixed pt.

Needed for IFT.

$f_a(h(a)) = h(a)$

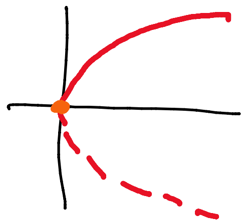
So $h(a)$ is the graph in the bifurcation diagram.

Proof: Let $G(a,x) = F(a,x) - x$. Thus, $G(a,x) = 0$

Proof: Let $G(a,x) = F(a,x) - x$. Thus, $G(a,x)=0$ if and only if $F(a,x) = x$. And $\frac{\partial G}{\partial x}(a,x) \neq 0$ if and only if $\frac{\partial F}{\partial x}(a,x) \neq 1$. Apply IFT to G. $\square$

4.1: Saddle-node bifurcation.

A bifurcation like the one seen in our previous example is one where 2 fixed points "approach" each other, then coincide and disappear. This is a saddle-node bifurcation.



This is a saddle-node bifurcation.

Aside: Recall differentiability classes. The class of continuous function is denoted by C^0, and we define C^k recursively. Namely, C^k is the class of functions whose derivatives are in the class of C^{k-1} functions. The class C^1 contains cont. diff., and C^2 contains functions whose derivatives are cont. diff. A function is smooth if it is in C^∞.

Goal: Understand

(1) descriptively

(2) mathematically

the common and fundamental bifurcations.

Thm (Saddle-node bifurcation)

Let $I, J \subseteq \mathbb{R}$ and $F: J \times I \to I$ be a C^2-function. Let $f_a(x) = F(a,x)$ for all $a \in J$, $x \in I$. Suppose $(\bar{a}, \bar{x}) \in J \times I$ such that:

1) $f_{\bar{a}}(\bar{x}) = F(\bar{a}, \bar{x}) = \bar{x}$,

2) $f'_{\bar{a}}(\bar{x}) = \frac{\partial F}{\partial x}(\bar{a}, \bar{x}) = 1$

2) $f'_{\bar a}(\bar x) = \frac{\partial F}{\partial x}(\bar a, \bar x) = 1$

3) $f''_{\bar a}(\bar x) = \frac{\partial^2 F}{\partial x^2}(\bar a, \bar x) \neq 0$,

4) $\frac{\partial F}{\partial a}(\bar a, \bar x) \neq 0$.

Then there exists an open interval K, containing $\bar x$, and a C^2-function $h: K \to \mathbb{R}$ s.t. $h(\bar x) = \bar a$ and $f_{h(x)}(x) = x$. Moreover, $h'(\bar x) = 0$ and $h''(\bar x) \neq 0$.

Proof: Consider the function $G(a,x) = F(a,x) - x$.

By (1), $G(\bar a, \bar x) = 0$. By (4)

$$\frac{\partial G}{\partial a} = \frac{\partial F}{\partial a} - \frac{\partial}{\partial a}(x) = \frac{\partial F}{\partial a},$$

so

$$\frac{\partial G}{\partial a}(\bar a, \bar x) \neq 0.$$

Apply IFT. There exists an interval K and a function $h: K \to \mathbb{R}$ s.t. $h(\bar x) = \bar a$ and for all $x \in K$, $G(h(x), x) = 0$. That is

$$F(h(x), x) - x = 0 \iff f_{h(x)}(x) = x.$$

Now we look at h' and h''. We implicitly differentiate $0 = G(h(x), x)$.

$$0 = \frac{d}{dx} G(h(x), x)$$

$$= \frac{\partial G}{\partial a}(h(x), x) \cdot h'(x) + \frac{\partial G}{\partial x}(h(x), x).$$

$$\left(= \frac{\partial G}{\partial a}\frac{da}{dx} + \frac{\partial G}{\partial x}\frac{dx}{dx} \right)$$

Solving for h' and setting $x = \bar x$,

$$h'(\bar x) = \frac{-\frac{\partial G}{\partial x}(\bar a, \bar x)}{\frac{\partial G}{\partial a}(\bar a, \bar x)} = \frac{-\left(\frac{\partial F}{\partial x}(\bar a, \bar x) - 1\right)}{\frac{\partial F}{\partial a}(\bar a, \bar x)} = \frac{0}{\frac{\partial F}{\partial a}(\bar a, \bar x)} = 0,$$

since the denominator is not 0.

Another implicit derivative + algebra:

$$h''(\bar x) = -\frac{\frac{\partial^2 F}{\partial x^2}(\bar a, \bar x)}{\frac{\partial F}{\partial a}(\bar a, \bar x)}. \qquad (*)$$

By (3), $h''(\bar x) \neq 0$. $\blacksquare$

by (3), $h'(\bar{x}) \neq 0$.

Case 1: $f''_{\bar{a}}(\bar{x}) < 0$. If $x < \bar{x}$, $f'_{h(x)}(x)$

Come back on Friday!