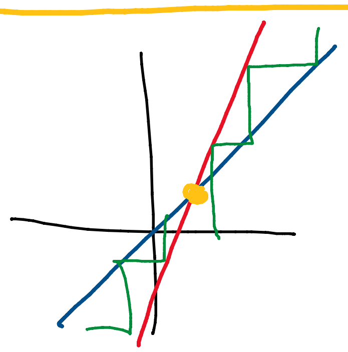
Lecture 7

Friday, May 15, 2020 8:18 AM

Ex 3.2 $x_{n+1} = ax_n + b$

Do work...

$$x_n = a^n\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}.$$



We discuss Sharkovsky's Theorem.
This was proved in the 1960's.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$1 < 2 < 3 < \ldots$

This is a total order: every pair of elements can be compared.

We will develop the Sharkovsky order, which is a diff total order on $\mathbb{N}$.

$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \ldots$ odd

$\ldots \prec 2\cdot3 \prec 2\cdot5 \prec 2\cdot7 \prec 2\cdot9 \prec 2\cdot11 \prec \ldots$ $2\cdot$odd

$\ldots \prec 2^2\cdot3 \prec 2^2\cdot5 \prec 2^2\cdot7 \prec 2^2\cdot9 \prec 2^2\cdot11 \prec \ldots$ 2^2odd

$\vdots$

$\ldots \prec 2^n\cdot3 \prec 2^n\cdot5 \prec 2^n\cdot7 \prec \ldots \ldots$ $2^n\cdot$odd

$\vdots$

$\ldots \prec 2^5 \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1.$

Ex: $3 \prec n$ for all $n \in \mathbb{N}$.

51 & 64 compare? $51 \prec 64$

10 & 8? $10 = 2\cdot5$, $8 = 2^3$

$10 \prec 8.$

For all $n \in \mathbb{N}$, $n \prec 1$.

$3 \prec n \prec 1$

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$3 \prec n \prec 1$.

<u>Theorem (Sharkovsky 1964).</u> Let $f: \mathbb{R} \to \mathbb{R}$ is continuous. If f has a periodic point of min. period p and $p \prec q$ (Sharkovsky order), then f has a periodic point of min. period q.

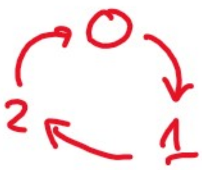
<u>Cor.</u> Let $f: \mathbb{R} \to \mathbb{R}$ be cont. If f has a periodic point of min. period 3, then f has a periodic point of min. per. k, for all $k \in \mathbb{N}$.

This is an existence proof not a construction.

<u>Ex: 3.14.</u> We create an example with a periodic point with min per. $\underline{3}$.

Let f be a poly of degree 2 such that

$$f(0) = 1$$
$$f(1) = 2$$
$$f(2) = 0.$$



$f(x) = \underline{(x-2)(ax+b)}$, for some $a, b \in \mathbb{R}$.

$$\underline{1} = f(0) = -2b \Rightarrow b = -1/2.$$
$$2 = f(1) = -(a - 1/2) \Rightarrow a = -3/2.$$

$$\boxed{f(x) = -\frac{1}{2}(x-2)(3x+1).}$$

We know there are periodic points of min. period 3. By Sharkovsky's theorem there are periodic points of min. period k, for all $k \in \mathbb{N}$.

$$f^k(x) = \underbrace{f(\cdots f(x))}_{k\text{-times}}$$

(Look at pictures in lecture notes.)

$$f^{\circ k}(x) = \underbrace{f(\cdots f(x))}_{k \text{ times.}}$$ (Look at pictures in lecture notes.)

We prove some special cases of Sharkovsky thm.

<u>Prop 3.15:</u> Let $f: I \to I$ be continuous. If f has a periodic point of min. per. 2, then f has a fixed point.

<u>proof:</u> Let x_0 be a periodic pt of m.p. 2. Let $x_1 = f(x_0)$, so $f(x_1) = f^2(x_0) = x_0$

$$x_0 \underset{f}{\rightleftarrows} x_1$$

These are diff. points so one is larger than the other. Let's say

<u>$x_0 < x_1$.</u> (normal order)

$$\boxed{\begin{array}{ll} x_0 < f(x_0) & (*) \\ f(x_1) < x_1 & (**) \end{array}}$$

From $(*)$ f must lie above the id function at x_0. From $(**)$, f must lie below the id function at x_1. By the Intermediate Value Theorem, $\exists\, \bar{x}$ in between x_0 and x_1 s.t. $\bar{x} = f(\bar{x})$. $\square$

<u>Cor. 3.16:</u> Let $f: \mathbb{R} \to \mathbb{R}$ be cont. If f has a per. pt. of m.p. 2^n, then f has a per pt. of m.p. 2^k, for all $0 \le k < n$.

<u>proof:</u> Consider the function

$$h_{n-1} := f^{2^{n-1}}.$$

Since f is cont., h_{n-1} is also cont. Since f has a <u>per. pt.</u> x of <u>min. per. 2^n</u>, then

$$\begin{aligned} h_{n-1}^2(x) &= \left(f^{2^{n-1}}\right)^2(x) \\ &= f^{2^{n-1}}\left(f^{2^{n-1}}(x)\right) \\ &= f^{2^n}(\underline{x}) \\ &= x. \end{aligned}$$

$$= f(x)$$
$$= x.$$

This shows that x has per. 2 under h_{n-1}. By Prop. 3.15, h_{n-1} has a fixed point, call it y:

$$f^{2^{n-1}}(y) = h_{n-1}(y) = y$$

This implies that y is a periodic point of period 2^{n-1}. Applying downward induction proves the statement. $\square$

Chapter 4: Bifurcations in 1-dim. systems.

Start with a family of examples.

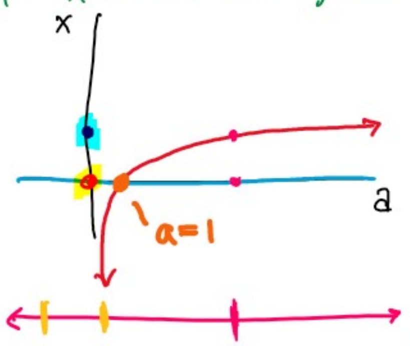
$$f_a : \mathbb{R} \to \mathbb{R} \quad f_a(x) = ax(1-x).$$

The value a is a parameter that can be adjusted. This gives us a family of logistic maps. We saw the fixed pts are

$$\boxed{x=0} \longleftrightarrow , \quad \boxed{x = \frac{a-1}{a}} \cdot (0<a)$$

How do the fixed points change in terms of how a changes?

$1 - \frac{1}{a}$



Not only do we care about # of fixed points (and where they occur), but we care about their properties.

What can happen is that as we "smoothly" change the values for a, we might witness a qualitative change in behavior of the fixed pts. This is what is known as a **bifurcation**.

change in behavior of the fixed pts.
This is what is known as a **bifurcation.**

Ex 4.1: $f_b(x) = -x^2 + x + b$. The fixed pts are:

$$x = \pm\sqrt{b}.$$

$f(x) = x$

$-x^2 + \cancel{x} + b = \cancel{x}$

$-x^2 + b = 0$

Case 1: $b < 0$. No fixed pts.

Case 2: $b = 0$. Exactly 1 fixed pt.

$f_b'(x) = -2x + 1, \quad f_b'(0) = 1.$

$f_0(x) = x - x^2 = x(1-x)$

This is the logistic for $a = 1$.
This fixed is unstable.

Case 3: $b > 0$. There is exactly 2 pts.

$f_b'(x) = -2x + 1$

Applying Thm 3.5:

$$|f_b'(\sqrt{b})| = |-2\sqrt{b} + 1|.$$

When $0 < b < 1$, then $|f_b'(\sqrt{b})| < 1$, so this is an attracting, stable fixed pt.
When $b > 1$, then $|f_b'(\sqrt{b})| > 1$, so this is a repelling unstable f.p.
For $x = -\sqrt{b}$, this is always unstable (Check!).