Lecture 6

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Thm 3.5: Let $f: I \to I$ be continuously differentiable, and suppose $x \in I$ is a fixed point.

(i) If $|f'(x)| < 1$, then x is attracting and stable.

(ii) If $|f'(x)| > 1$, then x is repelling and unstable.

Proof (ii). Choose λ s.t. $|f'(x)| > \lambda > 1$.

Since f' is continuous $\exists \delta > 0$ s.t. $\forall y \in B_\delta(x)$ $|f'(y)| > \lambda$. Again choose any $y \in B_\delta(x)$ arbitrary with $y \neq x$. By the Mean Value Theorem $\exists c$ inbetween x and y s.t.

$$\lambda < |f'(c)| = \frac{|f(x) - f(y)|}{|x-y|}.$$

so

$$|x - f(y)| = |f(x) - f(y)| > \boxed{\lambda |x-y|} > |x-y|.$$

If $\lambda|x-y| \geq \delta$, then $|x - f(y)| \geq \delta$. This implies that $f(y)$ is outside of $B_\delta(x)$, and we would be finished. Therefore suppose $f(y) \in B_\delta(x)$. By the MVT, $\exists c_1$ inbetween x and $f(y)$ s.t.

$$\lambda < |f'(c_1)| = \frac{|f(x) - f^2(y)|}{|x - f(y)|}$$

Again,

$$|x - f^2(y)| = |f(x) - f^2(y)| > \lambda|x - f(y)| > \lambda^2 |x-y|.$$

So in other words,

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$$\boxed{|x-f^2(y)| > \lambda^2 |x-y|}$$

Since $\lambda > 1$, eventually $\underbrace{\lambda^n}_{\text{exp}} \underbrace{|x-y|}_{\text{const.}} \geq \delta$.

If $|x-f^2(y)| < \delta$, then $f^2(y) \in B_\delta(x)$ and do above argument again. Then

$$|x-f^3(y)| > \lambda^3 |x-y|.$$

After some finite number m, $f^m(y) \notin B_\delta(x)$. Thus, x is a repelling fixed point. $\square$

Ex 3.6: $f_a(x) = ax(1-x)$ for $a \in (1,2)$.

The fixed points of f_a are $x=0$ and $x = \frac{a-1}{a}$.

$$f_a'(x) = a - 2ax$$

Then

$$|f_a'(0)| = |a| = a > 1$$

By Thm 3.5, $x=0$ is a repelling fixed point.

Then

$$\begin{aligned} \left|f_a'\left(\tfrac{a-1}{a}\right)\right| &= \left|a - 2a\left(\tfrac{a-1}{a}\right)\right| \\ &= |a - 2a + 2| \\ &= |2-a| \\ &< 1 \end{aligned}$$

Therefore, $x = \frac{a-1}{a}$ is an attracting fixed point.

Question: What happens when

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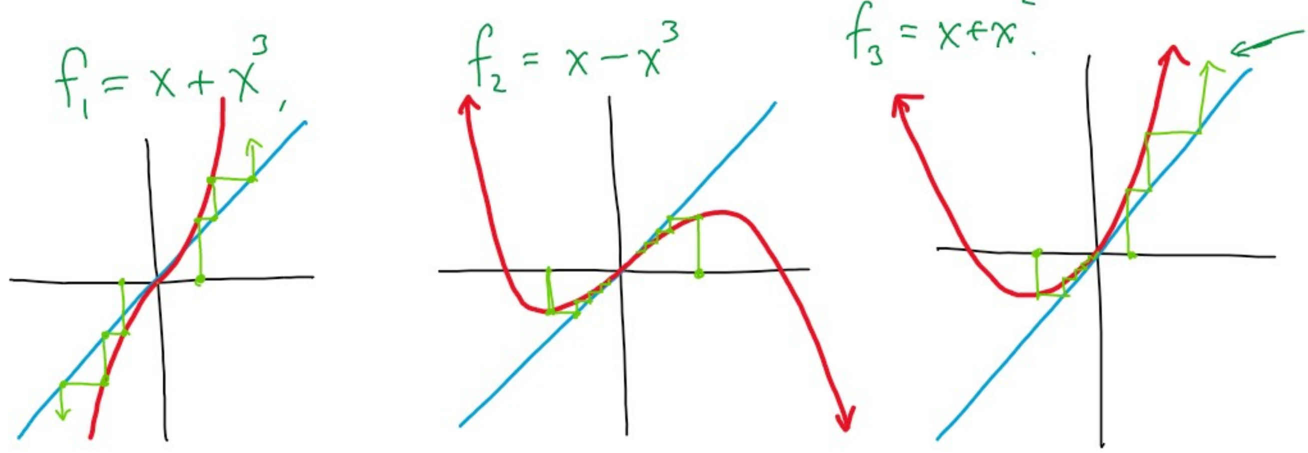
$$|f'(x)| = 1 \; ??$$



Prop 3.8: Let $f: \mathbb{R} \to \mathbb{R}$ be continuous with fixed point $x = 0$.

(i). If $\exists\, r > 0$ s.t. $\forall x \in B_r(0) \setminus \{0\}$
$|f(x)| < |x|$, then $x = 0$ is attracting.

(ii). If $\exists\, r > 0$ s.t. $\forall x \in B_r(0) \setminus \{0\}$
$|f(x)| > |x|$, then $x = 0$ is repelling.

Proof. (i) Fix $x_0 \in \mathbb{R}$, $x_{n+1} = f(x_n)$, $\forall n \in \mathbb{N}$.
Suppose $\forall x \in B_r(0) \setminus \{0\}$, $|f(x)| < |x|$. Thus,

$$|f(x)| < |x| < r$$

so $f(x) \in B_r(0)$. Because $|f(x)| < |x|$ for all $x \in B_r(0) \setminus \{0\}$, f cannot have <u>additional</u> fixed points in $B_r(0)$. In particular

$$|f(x)| = |x|$$

if and only $x = 0$.

Now $x_0 \in B_r(0)$,

$$r > |x_0| > |x_1| > |x_2| > \cdots$$

Now $x_0 \in I_r(c)$,

$$r > |x_0| > |x_1| > |x_2| > \cdots$$

Therefore, the sequence $(|x_n|)_{n=0}^{\infty}$ is mon. dec. and bounded. Thus, the sequence converges:

$$\bar{x} = \lim_{n\to\infty} |x_n| .$$

(Want $\bar{x} = 0$) If $\exists k \in \mathbb{N}$, where $|x_k| = 0$, then $x_k = 0$ and $\bar{x} = 0$. Now let's assume that $|x_n| > 0$, for $n \in \mathbb{N}$.

$$\rightarrow |x_1| > |x_2| > |x_3| > \cdots > |x_n| > \cdots > 0.$$

There exists a subsequence of $(|x_n|)_{n=0}^{\infty}$ where every x_{n_k} has the same sign: that is $\forall k \in \mathbb{N}$ x_{n_k} is all positive or all negative.

$$\rightarrow \underbrace{|x_{n_1}| > |x_{n_2}| > \cdots > |x_{n_k}| >}_{\text{all same sign.}} \cdots > 0.$$

There exists a (further) subsequence $(x_{n_{k_j}})_{j=1}^{\infty}$ s.t. $\forall j \in \mathbb{N}$ $f(x_{n_{k_j}})$ have the same sign. This is a subsequence of the original seq. so

$$\bar{x} = \lim_{j\to\infty} x_{n_{k_j}} .$$

For all $j \in \mathbb{N}$,

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$$|x_{n_{k_j}}| > |f(x_{n_{k_j}})| = |x_{n_{k_j}+1}|$$

$$|\bar{x}| \geq \lim_{j\to\infty} |f(x_{n_{k_j}})| = |\bar{x}|.$$

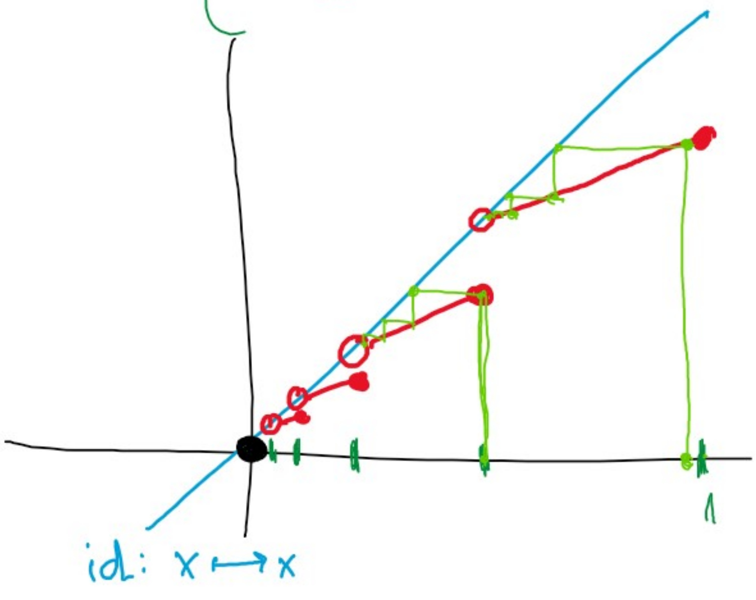
Thus, $|\bar{x}| = |f(\bar{x})|$, so $\bar{x} = 0$. Therefore $\lim_{n\to\infty} |x_n| = 0$. Since x_0 is arbitrary, $x=0$ is an attracting fixed point. $\square$

For (ii) prove like (i).

Question: Why continuity?

Ex: 3.9 $f : [0,1] \longrightarrow [0,1]$

$$f(x) = \begin{cases} \frac{x}{2} + \frac{1}{2^{k+1}} & \text{for } \frac{1}{2^k} < x \leq \frac{1}{2^{k-1}}, \\ 0 & x = 0. \end{cases}$$

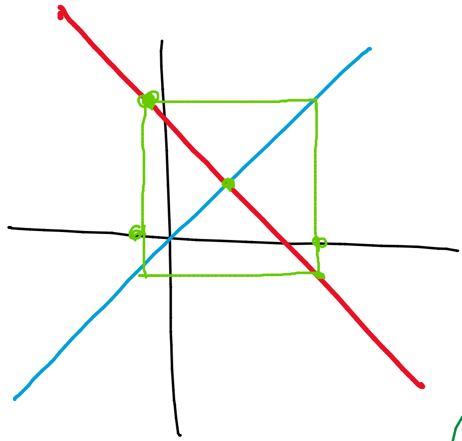


3.1: Periodic points and Sharkovsky's Thm.

Thm.

$$f(x) = ax + b \longrightarrow f(x) = b - x.$$

Fixed point at $x = b/2$.

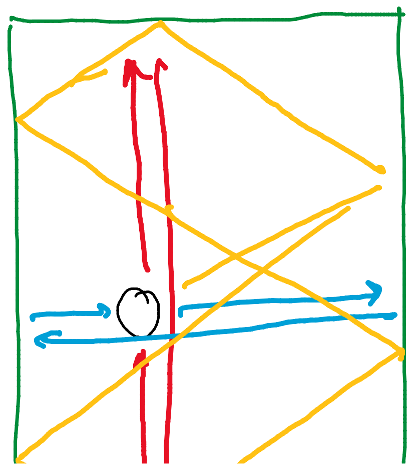


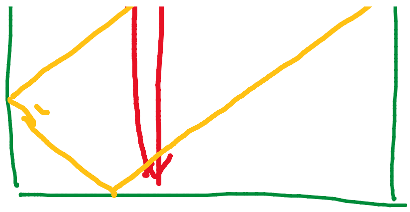
Have period 2, so these are fixed points of f^2.

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= f(b - x) \\ &= b - (b - x) \\ &= x. \end{aligned}$$

Thus, <u>every point</u> is periodic with period 2.

In general periodic points are hard to find.





Ex: $f(x) = x^2 - 1$

Fixed pts have to satisfy

$$x = x^2 - 1$$

$$0 = x^2 - x - 1$$

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

What about periodic points?

$$f^2(x) = f(x^2-1)$$
$$= (x^2-1)^2 - 1$$
$$= x^4 - 2x^2 + 1 - 1$$
$$= x^2(x^2-2)$$

Periodic points of period 2 satisfy

$$x = x^2(x^2-2)$$

$$0 = x(x^3 - 2x - 1)$$

$$= \underbrace{x(x+1)}_{\text{periodic pts.}}\underbrace{(2x-1+\sqrt{5})(2x-1-\sqrt{5})}_{\text{fixed points.}}$$

So $x=0$ and $x=-1$ are periodic points of min. period 2.

$$(-1, 0, -1, 0, -1, \dots)$$

Doing things this way is challenging.
In this example find periodic points of period p requires factoring a poly. of degree $2p$, which gets challenging.

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