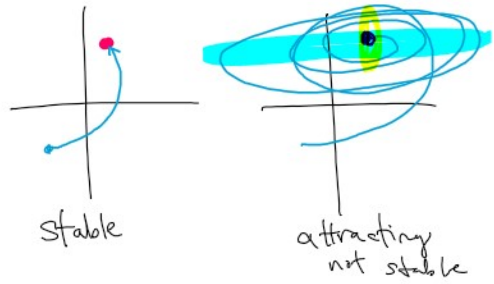


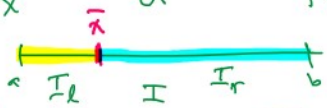
Prop. 2.24: If $f:\mathbb{R}\to\mathbb{R}$ is continuous with an attracting fixed pt. $\bar{x}$, then $\bar{x}$ is a stable fixed point of $x_{n+1}=f(x_n)$.

proof. By assumption $\bar{x}$ is an attractor. Let $I=(a,b)\subseteq\mathbb{R}$ containing $\bar{x}$ be the maximal interval s.t. $\forall x\in I$,

$$\boxed{\lim_{n\to\infty} f^n(x)=\bar{x}}$$

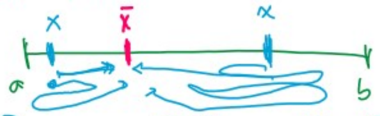
By construction, I cannot contain any other fixed points. So $\forall x\in I\setminus\{\bar{x}\}$, either

$$f(x)<x \quad \text{or} \quad f(x)>x.$$



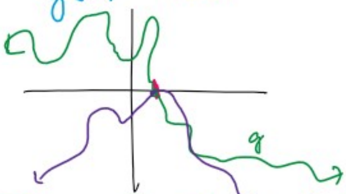
$I_l=(a,\bar{x})$, $\qquad I_r=(\bar{x},b)$.

Claim 1: For all $x\in I_l$, the orbit of x is monotonically inc., and $\forall x\in I_r$ the orbit of x is mon. dec.



proof of claim: Define $g:\mathbb{R}\to\mathbb{R}$ where $g(x)=f(x)-x$. Since f is continuous, so is g. The func. f cannot have periodic points in $I\setminus\{\bar{x}\}$; otherwise this would not converge to $\bar{x}$. This implies that g has exactly one root, namely $x=\bar{x}$,

$$g(\bar{x})=f(\bar{x})-\bar{x}=0.$$



Suppose $g>0$ on I_r. That is, $\forall x\in I_r$

$$f(x)>x.$$

For all $x\in I_r$ and $n\geq 1$

$$\longrightarrow \quad f^{n+1}(x)=f(f^n(x))>f^n(x)>x.$$

Thus, the seq $(f^n(x))_{n=0}^{\infty}$ is mono. inc.
Since $g>0$ on I_r, it follows that $x>\bar{x}$ $\forall x\in I_r$. This shows that $(f^n(x))$ does not converge to $\bar{x}$, a contradiction.
Hence, $\underline{g<0 \text{ on } I_r}$. (Similar for I_l). $\square$

$$x>f(x)$$

Hence, $\underline{g<0}$ on I_r. (Similar for I_ℓ). $\square$

$x > f(x)$

Back to the proof of Prop. 2.24.
Let $\varepsilon > 0$. Since f is continuous there exists $\delta > 0$ s.t. $\|x-\bar{x}\|<\delta$ implies

$$\|f(x)-\bar{x}\| < \varepsilon.$$

We can choose δ to be less than ε. Thus, $\delta \le \varepsilon$. Now, let $x \in I \setminus \{\bar{x}\}$ s.t. $\|x-\bar{x}\|<\delta$. Either $x\in I_r$ or $x \in I_\ell$.
The argument will be similar for both. We choose to work with $x \in I_r$. Thus,

$$\|x-\bar{x}\| = x-\bar{x} > 0.$$

From Claim 1 it follows that:

$$\rightarrow x > f(x) > f^2(x) > \cdots > \bar{x}.$$

For all $n \ge 0$, $\|f^n(x)-\bar{x}\|<\varepsilon$.

$$\begin{aligned}\|f^n(x)-\bar{x}\| &= f^n(x)-\bar{x} \\ &< x-\bar{x} \\ &= \|x-\bar{x}\| \\ &< \delta \le \varepsilon.\end{aligned}$$

$x \in I_\ell$. $= \bar{x} - f^n(x) < \bar{x} - x$

Therefore $\bar{x}$ is stable. $\blacksquare$

Ex: 2.25 in notes: good example of attracting but not stable.

2.3: Periodic Orbits

Def: Let x be a fixed pt. of $f: X \to X$. A point $y \in X$ is called **forward asymptotic** if

$$\lim_{n\to\infty} f^n(y) = x.$$

The collection of all forward asymptotic points is sometimes called a stable set.

Def: A periodic orbit $O^+(x)$ of period p is a **stable orbit** if each of its points

$$x,\ f(x),\ f^2(x), \dots$$

a **stable orbit** if each of the p...

$$x,\ f(x),\ f^2(x),\ \dots$$

is a stable fixed pt of f^p. A periodic orbit is **unstable** if it is not stable.

<u>Prop. 2.28:</u> Let $f: X \to X$ be continuous. Then a periodic orbit $O^+(x)$ of period p stable if and only if x is a stable fixed point of f^p.

<u>proof:</u> Proof in lecture notes.

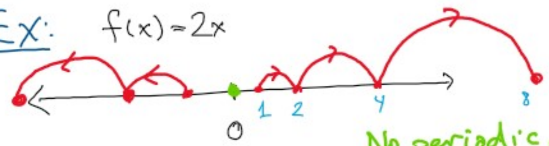
One can define <u>attracting/repelling</u> orbits in the same manner as above.

<u>Chapter 3:</u> One-dim. systems.

This chapter is focused on dynamical systems of the form $f: I \to I$, where $I \subseteq \mathbb{R}$.

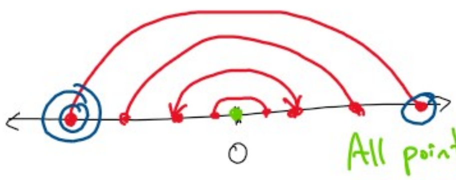
<u>Def</u>: A **phase portrait** is a graph of the possible "paths" of the system.

<u>Ex:</u> $f(x) = 2x$



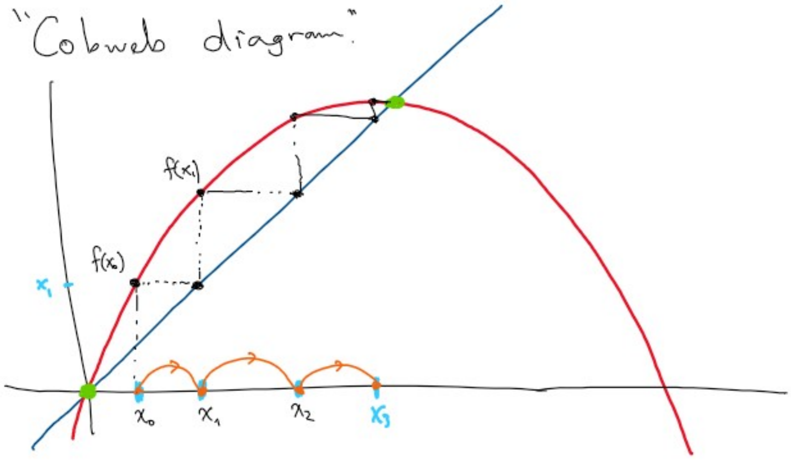
No periodic, but 1 fixed pt.

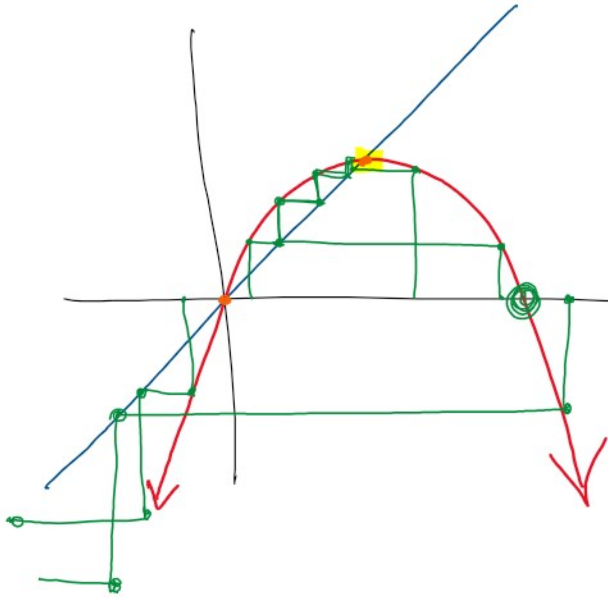
$f(x) = -x$



All points in $\mathbb{R}$ are periodic. All but $x=0$ have min. period 2.

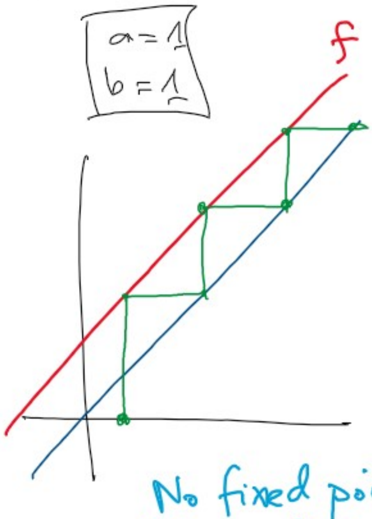
Ex: "Cobweb diagram"





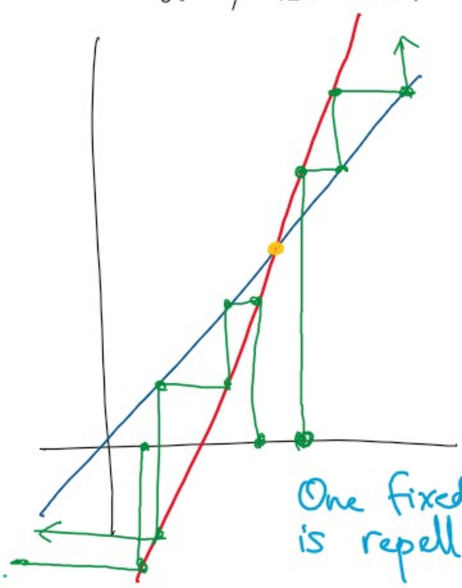
Ex 3.2: $f(x) = ax + b$

$a=1$
$b=1$



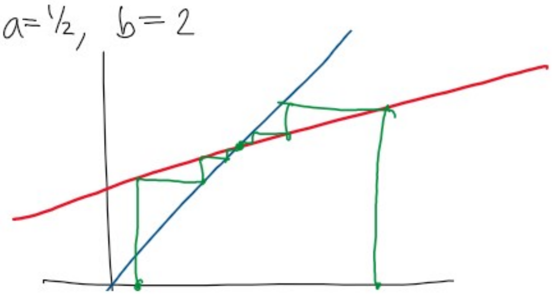
No fixed points. "Grows" indefinitely.

$a=2, \quad b=-2$.

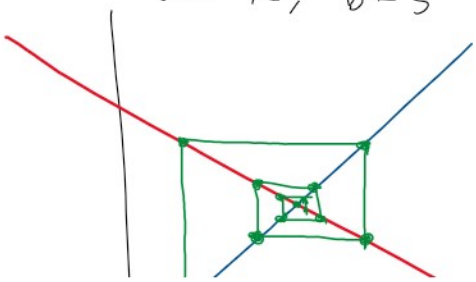


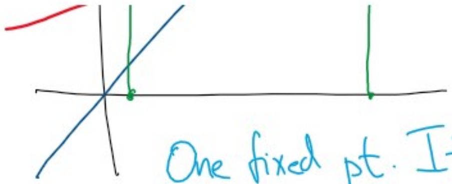
One fixed point. It is repelling.

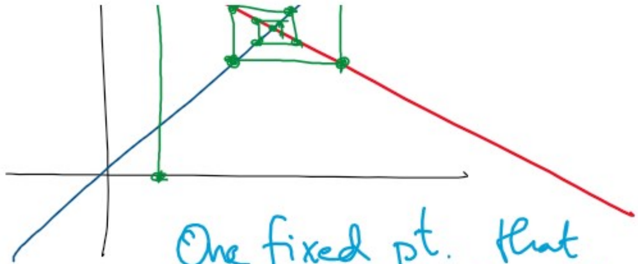
$a=1/2, \quad b=2$



$a=-1/2, \quad b=5$

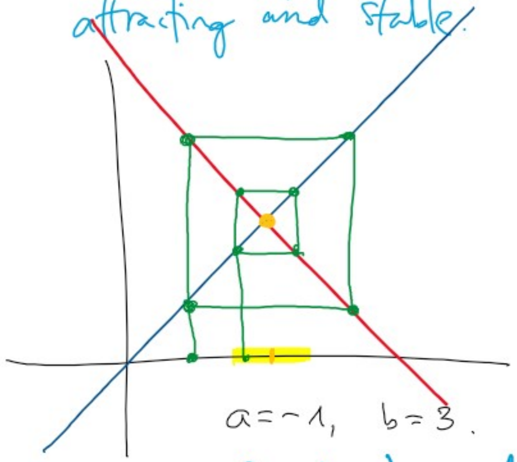






One fixed pt. It is attracting and stable.

One fixed pt. that is attracting and maybe st.



$a = -1, \quad b = 3.$

One fixed pt. and every other periodic with period 2.

This fixed is stable but not attracting
Not repelling also.

<u>Def:</u> Let $f: I \to I$ be continuously differentiable. A periodic point x of min. period p is hyperbolic if $|(f^p)'(x)| \neq 1$.

Recall, the <u>open</u> ball of radius $r \geq 0$ about a point $x \in \mathbb{R}$ is

$$B_r(x) = \{ y \in \mathbb{R} \mid |x-y| < r \}.$$

<u>Thm 3.5.</u> Let $f: I \to I$ be continuously differentiable, and suppose $x \in I$ is a fixed point of f.

(i) If $|f'(x)| < 1$, then x is attracting and stable.

(ii) If $|f'(x)| > 1$, then x is repelling and unstable.

<u>Proof:</u> (i) We assume $|f'(x)| < 1$. Find $\varepsilon = \lambda \in \mathbb{R}$, $|f'(x)| < \lambda < 1$. Since f' is continuous, $\exists \delta > 0$ s.t. $\forall y \in B_\delta(x)$

$$|f'(y) - 0| < \lambda = \varepsilon$$

$$|f'(y)| < \lambda.$$

δ Now fix a choice of $y \in B_\delta(x)$ with $y \neq x$. By the Mean Value Theorem there exists c in between x and y s.t.

$$\boxed{\lambda > |f'(c)|} = \frac{|f(x) - f(y)|}{|x-y|} \quad \text{MVT}$$

$c \in B_\delta(x)$

Therefore,

$c \in B_\delta(x)$

Therefore,

$$\underline{|x-f(y)|} = |f(x)-f(y)| < \lambda|x-y| < \underline{|x-y|}$$

This implies that $f(y) \in B_\delta(x)$. By MVT $\exists c_1$ between $f(y)$ and x

$$\lambda > |f'(c_1)| = \frac{|f(x)-f(f(y))|}{|x-f(y)|}.$$

Thus,

$$|x-f^2(y)| < \lambda|x-f(y)| < \lambda^2|x-y| < |x-y|.$$

By induction:

$$\boxed{|x-f^n(y)| < \underbrace{\boxed{\lambda^n}}_{\to 0}|x-y|.}$$

Thus, $f^n(y) \to x$. Since this holds for all $y \in B_\delta(x)$, x is attracting. By Prop. 2.24, x is stable.

(ii). Similar argument. $|f'(x)| > 1$, so choose $|f'(x)| > \lambda > 1$.