Lecture 4

Tuesday, May 5, 2020 2:17 PM

Prop 2.8: If $x \in X$ is periodic point of period m, then $\forall q \in \mathbb{N}$, $f^q(x)$ is a p.p. of period m.

Def: An orbit is periodic if all points are periodic

Def: A point $x \in X$ is eventually periodic if x is not periodic and $\exists m > 0$ s.t. $f^m(x)$ is periodic.

Ex: $f(x) = ax(1-x)$.

$x_0 = 1$. $(1, 0, 0, \dots)$

periodic.

For now $x \in \mathbb{R}$, and we use the Euclidean norm: $\|x\| = |x|$ (absolute value)

Def: A func. $f: X \to X$ is a contraction if $\exists K \in [0,1)$ s.t. $\forall x, y \in X$.

$$\|f(x) - f(y)\| \leq K \|x - y\|.$$

Distances contract.

Lemma 2.16 If a function $f: X \to X$ is a contraction, then f is continuous.

(Prove yourself.)

Thm 2.17 (Banach's Fixed-Point Thm)

If $f: X \to X$ is a contraction, with $X \subseteq \mathbb{R}^d$ closed, then f has a unique fixed point $\bar{x} \in X$ and the orbit of every $x \in X$ converges to $\bar{x}$.

proof Let $x_0 \in X$, and set $x_n = f^n(x_0)$ $\forall n \geq 1$

For each $n \in \mathbb{N}$,

$$x_n = \sum_{i=0}^{n-1} (x_{i+1} - x_i) + x_0.$$

$$= x_n \boxed{- x_{n-1} + x_{n-1}} \boxed{- x_{n-2}} \cdots$$

$$= \underbrace{x_n - x_{n-1}}_{n-1} + \underbrace{x_{n-1} - x_{n-2}}_{n-2} \cdots$$

Thus, $(x_n)_{n=0}^{\infty}$ converges if and only if

$$\left(\sum_{i=0}^{\infty} x_{i+1} - x_i\right)$$

converges. Since f is a contraction $\exists K \in [0,1)$ s.t.

$$\|x_{i+1} - x_i\| = \|f(x_i) - f(x_{i-1})\| \leq K \cdot \|x_i - x_{i-1}\|.$$

Continuing this argument, we get:

$$\boxed{\|x_{i+1} - x_i\| \leq K^i \|x_1 - x_0\|.}$$

$$\left\|\sum_{i=0}^{\infty}(x_{i+1} - x_i)\right\| \leq \sum_{i=0}^{\infty}\|x_{i+1} - x_i\|$$
$$\leq \sum_{i=0}^{\infty} K^i \cdot \|x_1 - x_0\|$$
$$= \|x_1 - x_0\| \cdot \underbrace{\boxed{\sum_{i=0}^{\infty} K^i}}_{<\infty}$$

geometric series & $K < 1$.

$$< \infty.$$

Therefore, by above, $(x_n)_{n=0}^{\infty}$ converges.

$$\bar{x} = \lim_{n\to\infty} x_n.$$

Since X is closed, it contains all limit points, so $\bar{x} \in X$. ($\mathbb{R}^d$ is complete)

Now we prove uniqueness:

$$\bar{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1})$$
$$= f\left(\lim_{n\to\infty} x_{n-1}\right)$$
$$= f(\bar{x}).$$

Continuity

Thus, $\bar{x}$ is a fixed point. Suppose $\bar{y}$ is

$= f(\bar{x})$.

Thus, $\bar{x}$ is a fixed point. Suppose $\bar{y}$ is another limit for a seq, so it is also fixed.

$$\|\bar{x}-\bar{y}\| = \|f(\bar{x}) - f(\bar{y})\|$$
$$\leq K \cdot \|\bar{x}-\bar{y}\|.$$

Well since $K \in [0,1)$, the only possibility is that $\|\bar{x}-\bar{y}\| = 0$. Thus, $\bar{x} = \bar{y}$. $\square$

2.2. Fixed points

Def 2.19.

Def: Let $f: X \to X$, and x is a fixed point.

1) The point $x \in X$ is a **stable fixed pt** if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. for $y \in X$

$$\|x-y\| < \delta \text{ implies } \|f^n(y) - x\| < \varepsilon$$

$f(x) = x$

for all $n \geq 0$.

2.) The point $x \in X$ is an **attracting fixed pt** if $\exists \delta > 0$ s.t. $\forall y \in X$,

$$\|x-y\| < \delta \text{ implies } \lim_{n\to\infty} f^n(y) = x.$$

3) The point $x \in X$ is **unstable** if x is not stable.

4. The point $x \in X$ is a **repelling fixed point** if $\exists \delta > 0$ s.t. for $y \in X$,

$$0 < \|x-y\| < \delta \text{ implies } \|f^m(y) - x\| > \delta$$

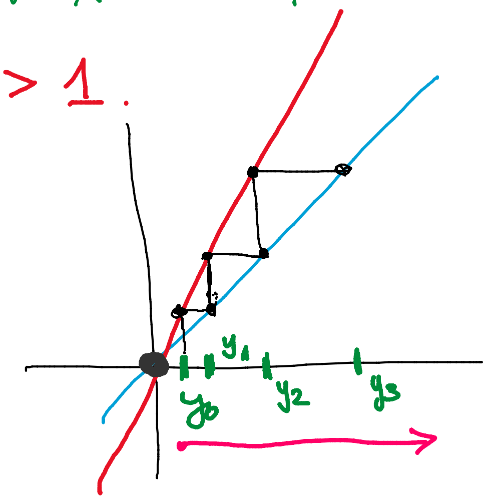
for some $m \geq 0$.

Ex: (2.20). $f(x) = ax$, define a system as

$$x_{n+1} = f(x_n).$$

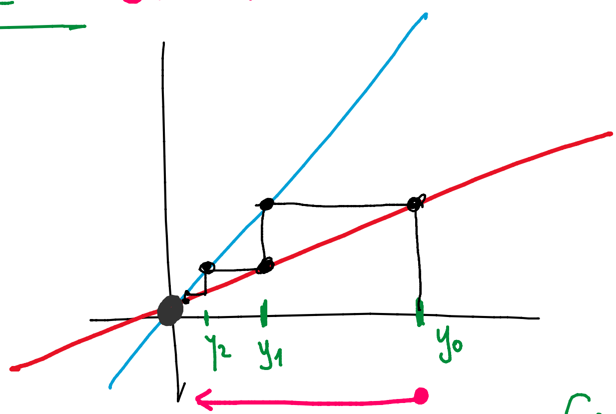
Note that $x=0$ is fixed.

Case 1: $a > 1$.



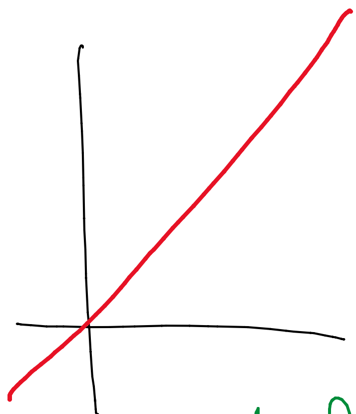
This is an unstable fixed and it is repelling.

Case 2: $a < 1$



This is a stable attracting fixed pt.

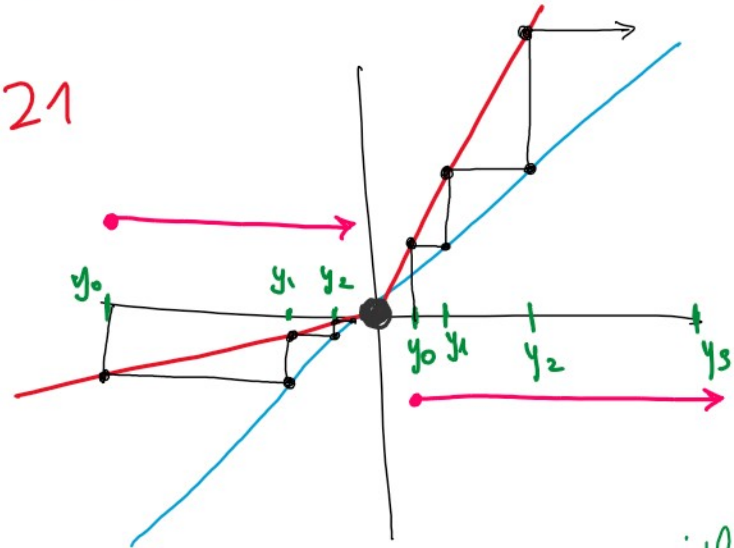
Case 3: $a = 1$



Every point is fixed. And every point is [illegible] but not attracting.

Every point is
is stable but not attracting.

Ex. 2.21



The fixed point $x=0$ is <u>neither</u> stable, attracting, nor repelling. Thus, just unstable.

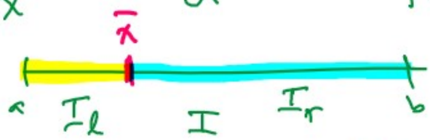
<u>**Prop. 2.24**</u>: If $f: \mathbb{R} \to \mathbb{R}$ is continuous with an attracting fixed pt. $\bar{x}$, then $\bar{x}$ is a stable fixed point of $x_{n+1} = f(x_n)$.

<u>proof.</u> By assumption $\bar{x}$ is an attractor. Let $I = (a,b) \subseteq \mathbb{R}$ containing $\bar{x}$ be the maximal interval s.t. $\forall x \in I$,

$$\boxed{\lim_{n\to\infty} f^n(x) = \bar{x}}$$

By construction, I cannot contain any other fixed points. So $\forall x \in I \setminus \{\bar{x}\}$, either

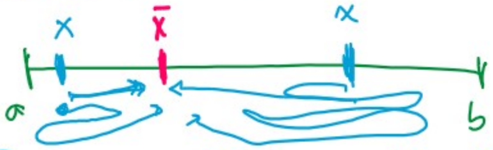
$$f(x) < x \qquad \text{or} \qquad f(x) > x.$$



$I_\ell = (a, \bar{x})$, $\qquad I_r = (\bar{x}, b)$.

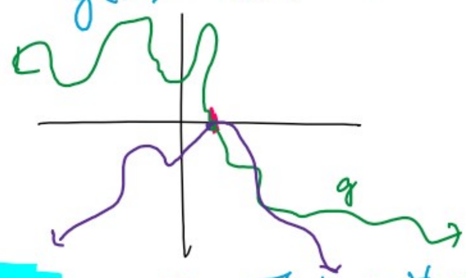
Claim 1: For all $x \in I_\ell$, the orbit of

Claim 1: For all $x \in I_\ell$, the orbit of x is monotonically inc., and $\forall x \in I_r$ the orbit of x is mon. dec.



proof of claim: Define $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = f(x) - x$. Since f is continuous, so is g. The func. f cannot have periodic points in $I \setminus \{\bar{x}\}$; otherwise this would not converge to $\bar{x}$. This implies that g has exactly one root, namely $x = \bar{x}$,

$$g(\bar{x}) = f(\bar{x}) - \bar{x} = 0.$$



Suppose $g > 0$ on I_r. That is, $\forall x \in I_r$

$$f(x) > x.$$

For all $x \in I_r$ and $n \geq 1$

$$\to \quad f^{n+1}(x) = f(f^n(x)) > f^n(x) > x.$$

Thus, the seq $(f^n(x))_{n=0}^{\infty}$ is mono. inc. Since $g > 0$ on I_r, it follows that $x > \bar{x}$ $\forall x \in I_r$. This shows that $(f^n(x))$ does not converge to $\bar{x}$, a contradiction. Hence, $g < 0$ on I_r. (Similar for I_ℓ). $\square$

$$x > f(x)$$