Lecture 3

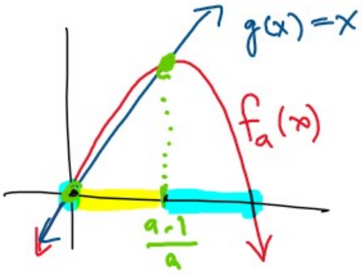
Tuesday, April 28, 2020 2:13 PM

$$x_{n+1} = 4x_n(1-x_n).$$

Suppose $a \in (1,2)$. $x_{n+1} = a x_n(1-x_n)$.

Let $f_a : \mathbb{R} \to \mathbb{R}$, where $f_a(x) = \boxed{ax(1-x) = x}$

$$ax(1-x) - x = 0$$



Note that the max of f_a is at $x = 1/2$, so for $x \in (0,1)$,

$$0 < f_a(x) \le f_a(1/2) = \frac{a}{4} < 1.$$

We have two fixed points:

$x = 0$ $\qquad$ $x = \frac{a-1}{a}$.

Consider $y_0 \in \left(0, \frac{a-1}{a}\right)$. We will show that

$$(y_n)_{n=0}^{\infty} = (y_0, y_1, y_2, y_3, \dots)$$

is monotonically increasing & $y_n \to \frac{a-1}{a}$

Lemma 1.8. For all $n \ge 0$,

$$\boxed{0 < y_n < \frac{a-1}{a}}.$$

Proof: For all $y \in (0,1)$, $0 < f_a(y) < 1$.

$$y_n = f_a(y_{n-1})$$

So $y_n > 0$, for all $n \ge 0$. Use induction.

Base case: $y_0 < \frac{a-1}{a}$, is clear.

Now by induction, we assume this is true for n, and we show it holds for $n+1$.

$$y_{n+1} = f_a(y_n)$$
$$< f_a\left(\frac{a-1}{a}\right)$$
$$= \frac{a-1}{a}$$

#1 $y_n < \frac{a-1}{a}$

#2 f is monotonically increasing on $\left(0, \frac{a-1}{a}\right)$.

Lemma 1.9. The seq. $(y_n)_{n=0}^{\infty}$ is mono. inc.

Proof. $y_{n+1} > y_n$?

$$y_{n+1} = f_a(y_n) = a y_n (1 - y_n)$$
$$> a y_n \boxed{\left(1 - \frac{a-1}{a}\right)}$$
$$= y_n. \qquad \square$$

$$\frac{a}{a} - \frac{a-1}{a} = \frac{1}{a}$$

Prop. 1.10 $\displaystyle \lim_{n\to\infty} y_n = \frac{a-1}{a}.$

Proof. Lemmas 1.8 & 1.9 show that $(y_n)_{n=0}^{\infty}$ is mon. inc. and bounded.

Thus, the limit exists.

$$\boxed{y_* = \lim_{n\to\infty} y_n.}$$

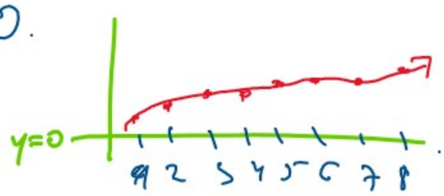
f_a is continuous.

$$y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \boxed{y_{n+1}} = \lim_{n\to\infty} f_a(y_n)$$
$$= f_a\left(\lim_{n\to\infty} y_n\right)$$
$$= f_a(y_*).$$

Thus, y_* is a fixed point! We have exactly two fixed points: $y=0$, $y=\frac{a-1}{a}$.

From Lemmas 1.8 & 1.9:

(y_n) is inc. and bounded below by $y=0$.



Thus, the only option for y_* is

$$\boxed{y_* = \frac{a-1}{a}} \qquad \square.$$

(This shows in part that $y=0$ is

(This shows in part that $y=0$ is unstable and $y=\frac{a-1}{a}$ is possibly stable.)

Chap 2: Discrete dynamical systems

$f, g: \mathbb{R}^d \longrightarrow \mathbb{R}^d \qquad (f \circ g)(x) = f(g(x))$.

The identity function $id: \mathbb{R}^d \to \mathbb{R}^d$ is the function defined by $x \mapsto x$.

Notation:
- $f^n(x) = (\underbrace{f \circ \cdots \circ f}_{n\text{-times}})(x) = f(f(\cdots(f(x))\cdots))$
- $f^0(x) = id(x) = x$.
- If f has inverse, then denote that by f^{-1}, so $\boxed{(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)}$

$$f^{-n}(x) = (\underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{n\text{-times}})(x)$$

- Note that $(f(x))^n \neq f^n(x)$.

Def 2.1: $X \subseteq \mathbb{R}^d$, $f: X \longrightarrow X$. The expression

$x_{n+1} - x_n = g(x_n)$ $\qquad x_{n+1} = f(x_n) \qquad (*)$

is called a difference equation. X is the state space. Any seq. $(x_n)_{n=0}^{\infty}$ satisfying $(*)$ is a solution. If $f(x_n)$ is independent of n, the system in $(*)$ is called a discrete autonomous dynamical system.

(All examples have been autonomous.)

Ex: (2.2.ii) $\boxed{x_{n+1} = (n+1)\, x_n} \implies \boxed{x_n = n!\, x_0}$

$$\boxed{x_{n+1} = (n+1)\, x_n} \implies \boxed{x_n = n!\, x_0}$$

not bad.

$$x_n = n x_{n-1} = n\cdot(n-1)\, x_{n-2}$$
$$= n(n-1)(n-2)\, x_{n-3}$$
$$\vdots$$
$$= n!\, x_0.$$

2.1: Discrete Autonomous Dynam Sys.

Throughout, $f: X \to X$, $X \subseteq \mathbb{R}^d$ and $x_{n+1} = f(x_n)$ is autonomous.

Def 2.4: Let $x \in X$.

(i) The seq

$$O^+(x) = (x, f(x), f^2(x), \dots)$$
$$= (x_0, x_1, x_2, \dots)$$

is the forward orbit (or sometimes orbit) of x relative to f.

most common.

(ii): If f is bijective, the seq

$$O^-(x) = (x, f^{-1}(x), f^{-2}(x), \dots)$$

is the backwards orbit.

(iii): Assuming bijective f, the seq

$$O(x) = (f^n(x))_{n=-\infty}^{\infty}$$
$$= (\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots)$$

is the full orbit.

Ex. 2.5 for various orbits.

Def: 2.6: $x \in X$ is a fixed point of $x_{n+1} = f(x_n)$ if $x = f(x)$.

- $x \in X$ is a periodic point of period m if
$$\boxed{f^m(x) = x}$$
(fixed pts are periodic!)
- If $p > 0$ is the smallest pos. int. such that $f^p(x) = x$, then p is the min. period

$f^p(x) = x$, then p is the min. period of x.

(Note: x has period m relative to f if and only if x is a fixed point of f^m.)

Lemma 2.7: If $x \in X$ is a periodic point of period m, then x has period nm for all $n \in \mathbb{N}$. Conversely, if x is a periodic point with min. period p and $f^m(x) = x$ for some $m > 0$, then $p \mid m$.

Proof: $\Longrightarrow$ Assume $x \in X$ is periodic with period m. We do this by induction. Base case $n=1$. That says, $f^m(x) = x$, which is true by def. Assume that $f^{mn}(x) = x$, and we will show that

$$f^{m(n+1)}(x) = x.$$

$$\begin{aligned} f^{m(n+1)}(x) &= f^{mn+m}(x) \\ &= f^{mn}(f^m(x)) \\ &= f^{mn}(x) \\ &= x. \end{aligned}$$

$f^2(x) = f(f(x))$

$\Longleftarrow$ Assume $x \in X$ with min. per. $= p$ and for some $m > 0$, $f^m(x) = x$. Use division alg. to write

$$m = q \cdot p + r, \quad \text{— remainder } \frac{m}{p}.$$

where $q, r \in \mathbb{Z}_{\geq 0}$ and $0 \leq r < p$.

$$\begin{aligned} x = f^m(x) &= f^{qp+r}(x) \\ &= f^r(f^{qp}(x)) \\ &= f^r(x) \end{aligned}$$

$\longrightarrow = f^r(x)$

Thus, $f^r(x) = x$, and $0 \le r < p$.

r=0: $f^r(x) = x$ means is $id(x) = x$.
This is always true.
$m = q \cdot p \implies p \mid m$. Done!

r>0: Well, $r < p$ and x has period r. This is a contradiction and cannot happen!

Thus, r must be 0, so $p \mid m$. $\square$