Lecture 2
Friday, 24 April, 2020 08:18

Continuous time exponential growth

$\boxed{x_{n+1} = a\, x_n}$ — discrete

$\boxed{x' = \lambda x}$

Want a function $x(t)$ that satisfies the diffi differential equation above.

$$\int \frac{x'}{x} dt = \int \lambda\, dt$$

$$\log|x| = \lambda t + C$$

$$\boxed{x(t) = De^{\lambda t}}$$

$t=0$, $x(0) = De^0 = D$. $D = x_0$.

$$\boxed{x(t) = x_0 e^{\lambda t}}$$

Diff. values of λ:

$\lambda = 0$: $e^{\lambda t} = 1$, $x(t) = x_0$.

$\lambda > 0$: If $x_0 = 0$, then $\boxed{x = 0}$ $\forall t$.

fixed point of the system.

$x_0 > 0$ $e^{\lambda t} \to \infty$ $x(t) \to \infty$.

$x_0 < 0$ " " $x(t) \to -\infty$

$$\underbrace{\qquad\qquad}_{t \to \infty}$$

$\lambda < 0$. $e^{\lambda t} \to 0$ as $t \to \infty$.

We still have fixed pt. $x = 0$.

- $x_0 > 0$, $x(t) \to 0$, as $t \to \infty$
- $x_0 < 0$ $x(t) \to 0$, as $t \to \infty$

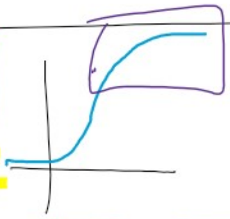
When $\lambda > 0$, $x = 0$ is unstable and repelling. When $\lambda < 0$, $x = 0$ is " " .

When $\lambda > 0$, $x = 0$ ... repelling. When $\lambda < 0$, $x = 0$ is stable and attracting.

Table 1.1 on p. 5.

1.3: Logistic Map

$$\boxed{x' = \boxed{\lambda x}\boxed{(1-x)}}$$



$x(0) = x_0 > 0$, $\lambda > 0$.

$$\frac{x'}{x(1-x)} = \lambda \Rightarrow \int \frac{x'}{x(1-x)}\,dt = \int \lambda\,dt.$$

partial frac. decomp. (bot p5)

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$$

$$\boxed{\begin{aligned}\int \frac{x'}{x(1-x)}\,dt &= \int \frac{x'}{x}\,dt + \int \frac{x'}{1-x}\,dt \\ &= \log|x| - \log|1-x|\end{aligned}}$$

$$\log|x| - \log|1-x| = \lambda t + C$$

$$\log\left|\frac{x}{1-x}\right| = \lambda t + C$$

$$\frac{x}{1-x} = De^{\lambda t}$$

$$\boxed{x(t) = \frac{De^{\lambda t}}{1 + De^{\lambda t}}}$$

$$x(0) = \boxed{\frac{D}{1+D} = x_0}$$

$$\boxed{x(t) = \frac{\left(\frac{x_0}{1-x_0}\right)e^{\lambda t}}{1 + \left(\frac{x_0}{1-x_0}\right)e^{\lambda t}}} = \frac{x_0 e^{\lambda t}}{1 - x_0 + x_0 e^{\lambda t}}$$

unstable Diff. values of x_0.

$x_0 = 0$: $D = 0$, then $x(t) = 0$ $\boxed{\forall t}$ for all

$0 < x_0 < 1$: $D > 0$. Thus $x(t) \to 1$, $t \to \infty$

fixed

$x_0 = 0$: $D = 0$, then $x(t) = 0$

$0 < x_0 < 1$: $D > 0$. Thus $x(t) \to 1$, $t \to \infty$

$$\frac{\boxed{De^{\lambda t} + 1} - 1}{1 + De^{\lambda t}} = \boxed{1 - \frac{1}{1 + De^{\lambda t}}}$$

$\frac{1}{1+De^{\lambda t}} \to 0$, $1 + De^{\lambda t} \to \infty$

$x_0 = 1$: Then $x(t) = 1$ $\forall t$.

stable.

$x_0 > 1$: $D < 0$ $x(t) \to 1$, $t \to \infty$.

Discrete:

$$x_{n+1} - x_n = \lambda x_n (1 - x_n).$$

$$\vdots$$

$$\boxed{y_{n+1} = a y_n (1 - y_n)}$$

This is much harder to analyze. Instead suppose $y_n = \sin^2(\psi_n)$ $\forall n$.

$$\begin{aligned}
\sin^2(\psi_{n+1}) &= y_{n+1} \\
&= a y_n (1 - y_n) \\
&= a \sin^2(\psi_n)(1 - \sin^2(\psi_n)) \\
&= a \sin^2(\psi_n) \cos^2(\psi_n). \\
&(a = 4 = 2^2) \\
&= 4 \sin^2(\psi_n) \cos^2(\psi_n) \\
&= \sin^2(2\psi_n)
\end{aligned}$$

$$\boxed{y_{n+1} = 4 y_n (1 - y_n)}$$

In summary:

$$\sin^2(\psi_{n+1}) = \sin^2(2\psi_n).$$

$$\boxed{y_n = \sin^2(2^n \psi_0)}$$

Closer look:

(fixed) $\psi_0 = 0$. $2^n \psi_0 = 0$. $y_n = 0$ $\forall n$.

Thus, $y = 0$ is fixed pt.

$\psi_0 = \frac{\pi}{2}$. $y_0 = \sin(\psi_0) = 1$.

$y_1 = \sin(2\psi_0) = 0$

$y_2 = \sin(2^2 \psi_0) = 0$

... fixed

$$y_2 = \sin(2^2 \psi_0) = 0$$

So $\psi_0 = \pi/2$ is eventually fixed.

$$\psi_0 = \frac{m\pi}{2^k}, \quad m, k \in \mathbb{Z}, \quad m \text{ is odd}$$
$$k > 0.$$

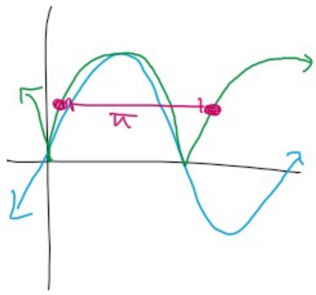
$$y_n = \sin^2\left(\frac{2^n m \pi}{2^k}\right)$$
$$= \sin^2\left(2^{n-k} m\pi\right)$$

When n gets larger than k,
Eventually $(n \to \infty)$ $y_n \to 0$.

$$\boxed{\psi_0 = \frac{m\pi}{k}}, \quad m, k \in \mathbb{Z}, \ k \text{ is odd.}$$

$$\boxed{y_n = \sin^2\left(\frac{2^n m\pi}{k}\right)}$$

$$\underbrace{\boxed{\sin^2(\theta) = \sin^2(\theta + \pi)}}_{\text{periodic function with period } = \pi.}$$



$$y_n = \sin^2\left(\frac{2^n m\pi}{k}\right) = \sin^2\left(\frac{2^n m \pi}{k} + l\pi\right) \quad l \in \mathbb{Z}.$$

Use division alg. $\frac{m}{k}$: $m = qk + r$
where $q \in \mathbb{Z}$ positive and $r \in \mathbb{Z}$
$$\boxed{0 \le r < k}$$

$$\frac{m}{k} = q + \frac{r}{k}$$

$$\sin^2\left(\frac{2^n m\pi}{k}\right) = \sin^2\left(\frac{2^n r\pi}{k} + 2^n q\pi\right)$$
$$= \sin^2\left(\frac{2^n r \pi}{k}\right)$$

Any $\frac{m}{k}$ reduces to just $\frac{r}{k}$, where
r is the remainder of m on k.

Ex 1.6: $\psi_0 = \frac{\pi}{3}$ $(m=1, k=3)$

n	remaidor $2^n m$ on k	y_n
0	1	3/4
1	2	3/4
2	1	3/4
3	2	3/4

1.7

Ex: $\psi_0 = \frac{2\pi}{11}$ $(m=2,\ k=11)$

n	remainder $2^n \cdot 2$ on 11	$y \cdot n$
0	2	0.29
1	4	0.83
2	8	0.57
3	5	0.98
4	10	0.08
5	9	0.29
6	7	
7	3	
8	6	
9	1	
10	2	

Significantly diff. from continuous case.