Lecture 19

Friday, June 26, 2020 8:29 AM

Ex. 7.17: Suppose $x_0 \in \mathbb{R}$ and

$$x' = x,$$
$$x(0) = x_0.$$

The solution is given by $x(t) = x_0 e^t$.

Want an analog of what we had in the discrete case: $x_0 \in \mathbb{R}$, then $x_{n+1} = f(x_n)$ gives a sequence of points from x_0 as "$t \to \infty$".

Define a function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\Phi(t, x_0) := \underline{x_0 e^t},$$

for $(t, x_0) \in \mathbb{R} \times \mathbb{R}$. The function Φ is cont. and it satisfies:

1. $\forall x_0 \in \mathbb{R}$,

$$\underline{\Phi(0, x_0) = x_0}$$ @ $t=0$, we are at the init. pt.

2. $\forall x_0, s, t \in \mathbb{R}$,

$$\begin{aligned} \Phi(s+t, x_0) &= x_0 e^{s+t} \\ &= \underline{x_0 e^t}\, e^s \\ &= \underline{\Phi(t, x_0)}\, e^s \\ &= \Phi(s, \Phi(t, x_0)). \\ &= \Phi(t, \Phi(s, x_0)). \end{aligned}$$

This defines a flow. $\boxtimes$

Def. Let $M \subseteq \mathbb{R}^d$. A continuous map $\Phi : \mathbb{R} \times M \to M$ is a *flow* if

1. $\forall x \in M$, $\Phi(0, x) = x$. time init. pt.

1. $\forall x \in M$, $\underline{\Phi}(\underset{\text{time}}{0}, \underset{\text{init. pt.}}{x}) = x$.

2. $\forall s, t \in \mathbb{R}$, $\forall x \in M$, $\underline{\Phi}(s+t, x) = \underline{\Phi}(s, \underline{\Phi}(t, x))$.

May see notation $\underline{\Phi}^t(x_0)$. (exponential notation)

If we have an autonomous system

$$x' = f(x),$$
$$x(0) = x_0,$$

where f is Lipschitz continuous, then we can define a flow from the solution.

Def: Let $M \subseteq \mathbb{R}^d$ and $\underline{\Phi}: \mathbb{R} \times M \to M$ a Flow. For $x_0 \in M$,

func. → 1. $x(t) = \underline{\Phi}(t, x_0)$ is the orbit through x_0,

set of outputs $\underline{\Phi}(t, x_0)$. → 2. The set $T(x_0) = T_{\underline{\Phi}}(x_0) = \{\underline{\Phi}(t, x_0) : t \in \mathbb{R}\}$ is the trajectory through x_0.

Ex: $x(t) = 2e^t$ is a solution to above example. This also defined a flow. Therefore, this is the orbit through $x_0 = 2$. The trajectory is

Trajectory of the orbit through $x_0 = 2$ → $T(2) \supseteq \{2, 2e, 2e^2, 2e^{-1}, \ldots$

$$\boxed{T(2) = (0, \infty)}$$

Def: Let $M \subseteq \mathbb{R}^d$ and $\underline{\Phi}: \mathbb{R} \times M \to M$ a flow.

1. A point $\bar{x} \in M$ is a fixed point if

$$\underline{\Phi}(t, \bar{x}) = \bar{x}$$

$$\Phi(t, \bar{x}) = \bar{x}$$

for all $t \in \mathbb{R}$.

2. A point $\bar{x} \in M$ is a periodic pt if $\exists \bar{t} > 0$ such that

$$\Phi(\bar{t}, \bar{x}) = \bar{x}.$$

The period of $\bar{x}$ is the smallest positive $\bar{t}$.

Recall, if $x' = f(x)$ and $\bar{x}$ is a fixed pt of this system, then $f(\bar{x}) = 0$. A flow is a solution: thus,

$$\underbrace{\frac{d\Phi}{dt}(t, \bar{x}) = \frac{d\bar{x}}{dt}}_{\text{Because } \Phi \text{ is a solution to system}} = 0.$$

Thus, $\Phi(t, \bar{x}) = \bar{x}$ $(\forall t \in \mathbb{R})$ and $f(\bar{x}) = 0$ really are equivalent.

Def: Let $M \subseteq \mathbb{R}^d$, $\Phi : \mathbb{R} \times M \to M$ is a flow. Assume $\bar{x} \in M$ is a fixed pt.

1. The pt $\bar{x}$ is Lyapunov stable if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\|\bar{x} - y\| < \delta$ implies that $\forall t > 0$

$$\|\underbrace{\Phi(t, \bar{x})}_{= \bar{x}} - \Phi(t, y)\| < \varepsilon.$$

Compare with discrete case: almost identical

2. The point $\bar{x}$ is asymptotically stable

2. The point $\bar{x}$ is asymptotically stable if $\bar{x}$ is stable and $\exists\, \delta > 0$ s.t. $\|\bar{x} - y\| < \delta$ implies

$$\lim_{t \to \infty} \Phi(t, y) = \bar{x}.$$

Compare with attracting fixed pt.

Ex: Fix $x_0 \in \mathbb{R}$ and consider

$$x' = -x, \qquad \longrightarrow f(x) = -x$$
$$x(0) = x_0.$$

The solution is

$$x(t) = x_0 e^{-t} =: \Phi(t, x_0).$$

Notice the point $x_0 = 0$ is a fixed point:

$$\Phi(t, 0) = 0, \qquad f(0) = 0.$$

Thus, for $y \in \mathbb{R}$ and $t \geq 0$: $\qquad |0 - y| = |y|$

$$
\begin{aligned}
|\Phi(t,0) - \Phi(t,y)| &= |y e^{-t}| \\
&= |e^{-t}| \cdot |y| \\
&= e^{-t} \cdot |y| \\
&\leq 1 \cdot |y| \\
&= |y|.
\end{aligned}
$$

Thus, $|\Phi(t,0) - \Phi(t,y)| \leq |y|$.

For every $\varepsilon > 0$, choose $\delta = \varepsilon$ so that

$$|y| < \delta$$

implies that

$$|\Phi(t,0) - \Phi(t,y)| \leq |y| < \delta = \varepsilon.$$

$$|\Phi(t,0) - \Phi(t,y)| \le |y| < \delta = \varepsilon.$$

Thus, $x=0$ is a stable fixed point. For asymptotically stable, note that

$$\Phi(t,y) = y \cdot e^{-t}.$$

As $t \to \infty$, $\Phi(t,y) \to 0$. Thus, $x=0$ is asymptotically stable.

Ch 8: Linear ODEs of higher order.

This is essentially linear algebra. We look at systems of the form:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned}$$

Now we write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

Thus, our ODE system is $x' = Ax$.

Remember, when $A = (a)$ (so 1-dimension)

$$x' = ax.$$

$$x(t) = C e^{at} = Ce^{At}$$

Actually, this always works for higher dimensions!

We will show that $x(t) = Ce^{At}$ is the solution for arbitrary $n \times n$ matrices A, but first, we define e^A.