Lecture 18

Tuesday, June 23, 2020 2:14 PM

Lipschitz:

Recall that $f: \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ satisfies the Lipschitz condition on Ω if $\exists L > 0$ s.t. $\forall (t,x), (t,y) \in \Omega$

$$\|f(t,x) - f(t,y)\| \leq L \|x-y\|.$$

This gives us conditions for uniqueness.

Thm 7.9: Suppose $f: \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is cont. and the following is an IVP:

system $\quad x'(t) = f(t, x(t)),$

init. cond. $\quad x(t_0) = x_0.$

If there exists $a, b \in \mathbb{R}$ s.t. f satisfies the Lipschitz condition on

$$\Omega = \{(t,x) \in \mathbb{R}\times\mathbb{R}^d \mid |t-t_0| \leq a, \|x-x_0\| \leq b\}$$

then there exists $c \in \mathbb{R}$ s.t. the IVP has a unique solution for all t in the interval:

$$I_c = \{t \in \mathbb{R} \mid |t-t_0| \leq c\}.$$

Unfortunately, only states that c exists, and tells us nothing about what it is equal to.

Cor 7.11 If $f: \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous on $\mathbb{R}\times\mathbb{R}^d$, then for every $t_0 \in \mathbb{R}$ the IVP,

$$x' = f(t, x(t))$$

$$x(t_0) = x_0,$$

has a unique solution for all $x_0 \in \mathbb{R}^d$.

Ex. 7.10:

$$\begin{cases} x' = tx^2, \\ x(0) = 1. \end{cases}$$

We want to find the solution and understand if it is unique. Define $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ where $f(t,x) = tx^2$. Find Ω s.t. f satisfies the Lipschitz cond. on Ω.

where $f(t,x) \in C^1$ ... Ω ...
the Lipschitz cond. on Ω.

$$\|f(t,x) - f(t,y)\| \leq L\,\|x-y\|.$$

if $x \in \mathbb{R}$, $\|x\| = |x|$

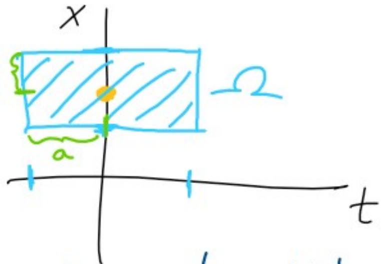
$$= \| tx^2 - ty^2 \|$$
$$= |t|\,|x^2 - y^2|$$
$$= \boxed{|t| \cdot |x+y|}\,|x-y|$$

Idea: bound the factor $|t| \cdot |x+y|$. That is, if $|t| \cdot |x+y|$ is bounded, then any L s.t. $L \geq |t| \cdot |x+y|$ works.

This means that both t and x need to be bounded in Ω. For example choose any $a, b \in \mathbb{R}_{>0}$, then

$$\Omega = \{ (t,x) \in \mathbb{R} \times \mathbb{R} \mid |t| \leq a,\ |x-1| \leq b \}.$$

On this set, both x and t are bounded.



The function f is Lipschitz cont. on all such Ω. To solve, we can just separate:

$$x' = tx^2$$

$\boxed{x(t)}$

$$\int \frac{x'}{x^2}\,dt = \int t\,dt$$

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{f'}{f}\,dx$$

$$-\frac{1}{x} = \frac{1}{2}t^2 + C$$

By init. cond. $C = -1$

$x(0) = 1$.

$$-\frac{1}{1} = \frac{1}{2}0^2 + C$$

$\boxed{C = -1}$

$$\boxed{-\frac{1}{x}} = \frac{1}{2}t^2 - 1$$

$$\frac{1}{x} = 1 - \frac{1}{2}t^2$$

$$x = \frac{1}{1 - \frac{1}{2}t^2} \cdot \frac{2}{2}$$

$$x = \frac{2}{2 - t^2}$$

Do a little algebra to rearrange; get

$$\boxed{x(t) = \frac{2}{2 - t^2}}$$

This is defined on $|t - 0| < \sqrt{2}$. Whatever I_c is equal to, it is inside $(-\sqrt{2}, \sqrt{2})$. ∎

7.2: Fixed pts and autonomous systems.

It is equal ... is inside $(-\sqrt{2}, \sqrt{2})$. ✗

7.2: Fixed pts. and autonomous systems.

Like in discrete case, fixed points tell a lot of inform about the system's behavior.

Def. Let $f: \mathbb{R}^d \to \mathbb{R}^d$ where $x' = f(x)$. A point $\bar{x} \in \mathbb{R}^d$ is a **fixed point** if $f(\bar{x}) = 0$.

Compare with discrete:

$x_{n+1} = f(x_n)$ $\quad f(\bar{x}) = \bar{x}$ $\quad$ iterating

$x' = f(x)$ $\quad f(\bar{x}) = 0$ $\quad$ derivative

These are the same, but look diff. because of context.

Ideas discussed for the discrete case carry over to continuous case.

Prop 7.13: Suppose $f: \mathbb{R} \to \mathbb{R}$ is cont. and $x' = f(x)$ is an autonomous system with orbit $x(t)$. If $\exists \bar{x} \in \mathbb{R}$ s.t.

$$\lim_{t \to \infty} x(t) = \bar{x},$$

then $\bar{x}$ is a fixed point of the system.

Proof. In notes. Read to check understanding.

$\boxed{\text{System: } x' = f(x)}$

Prop 7.14 Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz continuous on $\mathbb{R}^d$, and let $\bar{x}$ be a fixed pt. If $x(t)$ is an orbit of the system that contains $\bar{x}$, then the orbit is constant. In other words:

$$x(t) = \bar{x}.$$

proof. Since $x(t)$ is an orbit of the system, it is a solution to the following IVP. Since $\bar{x}$ is in the orbit, there is some $\bar{t} \in \mathbb{R}$

s.t.

$$IVP = \begin{cases} x' = f(x), \\ x(\bar{t}) = \bar{x}. \end{cases}$$

Now define a new function $\tilde{x}: \mathbb{R} \to \mathbb{R}^d$ where $\tilde{x}(t) = \bar{x}$ for all $t \in \mathbb{R}$. For $t \in \mathbb{R}$,

because $\bar{x}$ is fixed

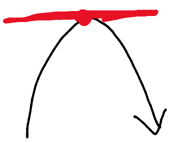
$$\tilde{x}'(t) = 0 = f(\bar{x}) = f(\tilde{x}(t))$$

same

Thus, $\tilde{x}' = f(\tilde{x})$, so $\tilde{x}$ is a solution to the IVP. Because f is Lipschitz continuous on $\mathbb{R}^d$, Theorem 7.9 implies that any solution is unique (so only one solution). We showed explicitly that $\tilde{x}$ is a solution, so $x(t) = \tilde{x}(t) = \bar{x}$. ∎

Prop 7.15: If $f: \mathbb{R} \to \mathbb{R}$ is L.c. on $\mathbb{R}$, then every orbit of the system is either constant or monotone.

proof: Suppose $x(t)$ is an orbit that is not monotone. Because not monotone, $\exists t_0 \in \mathbb{R}$ s.t.



$$0 = x'(t_0) = f(x(t_0)).$$

not monotone

x is an orbit

This implies $f(x(t_0)) = 0$, so $x(t_0)$ is a fixed point. Since f is L.c., Prop. 7.14 implies that the orbit must be constant. ∎

7.3: Flows

Ex: $x' = \sin x$. You could solve this:

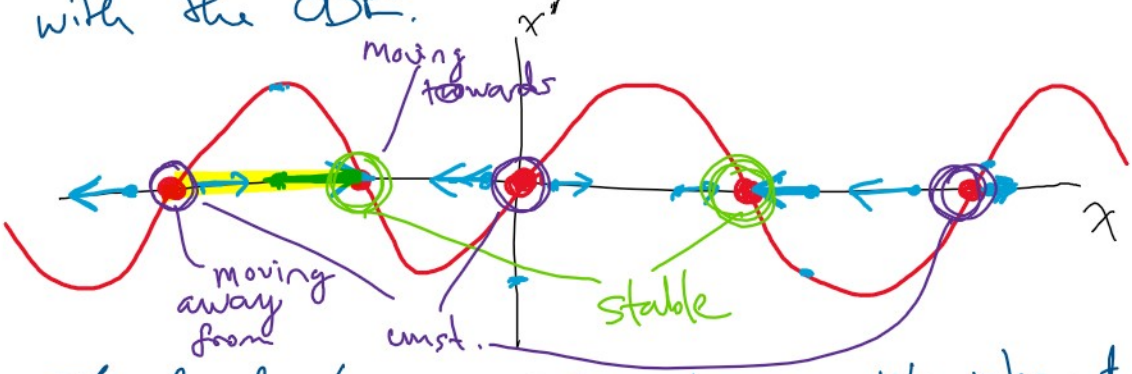
$$\boxed{t = \log \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|}.$$

csc ($\sin x$, $\cos x$, $\tan x$, $\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\cot x = \frac{1}{\tan x}$.)

$\sin^{-1}$ → ($\sin^{-1}(x)$: called $\arcsin(x)$. This is the inverse function so $\sin(\arcsin(x)) = x$.)

This solution is too complicated for me to understand. Sometimes we cannot even write down a solution. How can we understand the system?

Instead of working with the solution, we work with the ODE.



The fixed pts are easy to see. We interpret t as time, x is the position in $\mathbb{R}$, and x' as the velocity. Then $x' = \sin(x)$ this is a vector field. Negative derivatives go "backwards" and positive go "forward."