Lecture 17

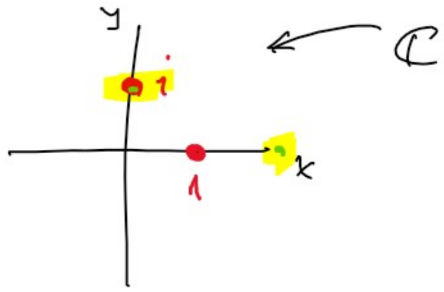
Friday, June 19, 2020 8:19 AM

With complex: $z = a + ib$, $|z| = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2+b^2}$.

$a, b \in \mathbb{R}$

✓ $|i| < |2|$.

✗ $i < 2$



If $\lambda \in \mathbb{C}$, then $|\lambda|$ = Euclidean distance from the origin.

Last time we saw sensitive dependence in the logistic model (for $a=4$).

<u>Def:</u> Let $f: X \to X$. We say f is **topologically mixing** if for every pair of nonempty open sets $U, V \subseteq X$, there exists $N \in \mathbb{N}$ s.t. $\forall n > N$, $f^n(U)$ has a nonempty interrection with V.

If X is an interval, then ↑ means that for every pair of nonempty open sets $U, V \subseteq X$ no matter how small, eventually

$$f^n(U) \cap V \neq \emptyset.$$

Choose any set.

Choose any set.

Recall $Y \subseteq X$ is <u>dense</u> in X if $\forall \varepsilon > 0$ and $\forall x \in X$, every ball $B_\varepsilon(x)$ contains elements of Y. (or $B_\varepsilon(x) \cap Y \neq \emptyset$.)

<u>Def:</u> $f: X \to X$. We say f is **chaotic** if
1. f has topological mixing.
2. the periodic points of f are dense in X.

<u>Fact:</u> $f = 4x(1-x)$ is chaotic on $[0,1]$.

The Lyapunov numbers do a good job of quantifying chaos.

<u>Ch 7:</u> Ordinary Diff. Eqs and Flows.

Moving to continuous time dynamical systems, and to ODEs (ordinary diff. eqs.).

An ODE is an equation involving derivatives.

$$\boxed{x' = x + x^2 t.} \qquad \text{(assuming } x(t)\text{)}$$

We say that

$$\frac{dx}{dt} =: x' = f(t, x)$$

is a <u>first order ODE</u>. If $x' = g(x)$, then this is an <u>autonomous ODE</u>.

An ODE of order n:

$$\frac{d^n x}{dt^n} =: x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)}).$$

n^{th} deriv.

Ex: An ODE of order 2:

$$x'' = px' + qx.$$

A system of ODEs contains 2 or more interconnected ODEs.

$$\left.\begin{aligned} x' &= x + t \\ y' &= 2y. \end{aligned}\right\} \text{separate.}$$

Typically, a system has the form:

$$\begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_d^{(n)} \end{pmatrix} = \begin{pmatrix} f_1(t, x, x', \dots, x^{(n-1)}) \\ f_2(t, x, x', \dots, x^{(n-1)}) \\ \vdots \\ f_d(t, x, x', \dots, x^{(n-1)}) \end{pmatrix}$$

$$\underline{x}^{(n)} = F(t, \underline{x}, \underline{x}', \dots, \underline{x}^{(n-1)}).$$

all vectors of dim. d.

$x = (x_1, x_2, \dots, x_d)$

$x' = (x_1', x_2', \dots, x_d')$ and so on.

Any higher order ODE can be rewritten as a system (of ODEs) of first order.

(just 1 ODE) $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1, x_2, \dots, x_{n-1}) \end{pmatrix}$$

idea: $x_1 = x$, $x_2 = x'$, $x_3 = x''$, and so on.

Notation: We will just write

$$x'(t) = f(t, x).$$

to be a first order system.

Ex 7.1:

$$x'' = \underbrace{\frac{3}{t} x' - \frac{4}{t^2} x + t}.$$

$x_1 = x$, $x_2 = x'$

system $\begin{cases} x_1' = x_2 \\ x_2' = \frac{3}{t} x_2 - \frac{4}{t^2} x_1 + t \end{cases}$.

First order systems (of ODEs) are perfectly general systems of ODEs.

Ex 7.2 and 7.3.

7.1: Existence and Uniqueness of solutions

7.1: [illegible] of solutions.

Suppose you have an ODE system that you care about. We develop a model and use it to understand behavior of the system. Regardless of our mathematical technology, a solution in reality exists.

If we can get a solution (somehow) we want to know that our solution will reflect what will happen in reality.

How can we be certain? One way, the solution is provably unique.

Showing uniqueness tends to involve more abstraction.

Ex: Consider the initial value problem (IVP):

$$\begin{cases} x' = 3x^{2/3} \\ x(0) = 0. \end{cases}$$

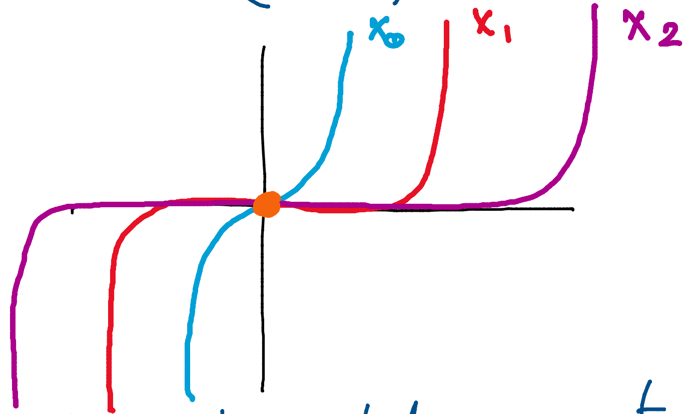
A solution is $x(t) = t^3$. This is separable. We have implicitly assumed that $x \neq 0$.
the zero function

Let $\alpha \geq 0$, and define

$$X_\alpha(t) = \begin{cases} (t+\alpha)^3 & t < -\alpha, \\ 0 & -\alpha \leq t \leq \alpha, \\ (t-\alpha)^3 & \alpha < t. \end{cases}$$

x_0 x_1 x_2

$(t-\alpha)$ $\qquad \alpha < t.$



This illustrates what we want to avoid in applications. We do not want multiple solutions to an IVP. $\square$

<u>Def:</u> A function $f:\underbrace{\mathbb{R}}_{t}\times\underbrace{\mathbb{R}^d}_{x}\longrightarrow\mathbb{R}^d$ is said to satisfy the <u>Lipschitz condition</u> on a set $\Omega\subseteq\mathbb{R}\times\mathbb{R}^d$ if $\exists$ a constant $L>0$ s.t. for all $(t,x),(t,y)\in\Omega$

$$\|f(t,x)-f(t,y)\|\leq L\,\|x-y\|.$$

The function is sometimes called Lipschitz continuous on Ω if it satisfies the Lipschitz condition on Ω.

<u>Ex 7.6:</u> $f(t,x)=t^2x^2$

(Underlying ODE is $x'=t^2x^2$.)

Since for all $x,y,t\in\mathbb{R}$ s.t. $x,y,t\in B_1(0)$,

$$|f(t,x)-f(t,y)|=|t^2(x^2-y^2)|$$
$$=\underbrace{|t^2|}_{\leq 1}\cdot|x^2-y^2|$$
$$\leq |x^2-y^2|$$

$$
\begin{aligned}
&\le |x^2 - y^2| \\
&= |x-y| \cdot \underbrace{|x+y|}_{\le 2} \\
&\quad L > 0 \\
&\le 2\,|x-y|.
\end{aligned}
$$

That is, $|f(t,x) - f(t,y)| \le 2|x-y|$. We can conclude that f satisfies the Lipschitz condition on

$$\Omega = \{(t,x) : |t| < 1,\ |x| < 1\}.$$