Lecture 16

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If $A = \begin{pmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix}$, — anything above diagonal.

then the eigenvalues of A are $a_1, a_2, \dots, a_n$.

$$A - \lambda I = \begin{pmatrix} a_1-\lambda & & * \\ & a_2-\lambda & \\ 0 & & \ddots \; a_n-\lambda \end{pmatrix}$$

The determinant of an upper triangular matrix (or lower) is the product of diagonal entries.

$$\det \begin{vmatrix} a_1 & & * \\ & a_2 & \\ 0 & \ddots & a_n \end{vmatrix} = a_1 \cdot a_2 \cdots a_n.$$

Def: $X \subseteq \mathbb{R}^d$, $f: X \to X$. Let $x_0 \in X$ and suppose $\rho(\underbrace{Df(x_n)}_{\text{Jacobian (at } x_n)}) > 0$, for all $n \in \mathbb{N}$.

$$\rho(\text{matrix}) = \max_{\lambda \text{ evals}} |\lambda| .$$

The Lyapunov number is

$$N(x_0) = \lim_{n\to\infty} \left(\rho(Df(x_n))\right)^{1/n}.$$

The Lyapunove exponent is

$$\Lambda(x_0) = \log N(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \rho(Df(x_n)).$$

The Lyapunov exponent is used to estimate stability. Let $x_0, y_0 \in X$ assume they are "close" together. By a Taylor expansion,

$$\underbrace{x_n - y_n}_{\text{"dist" at } n} = f^n(x_0) - f^n(y_0) \approx \underbrace{Df^n(x_0)}_{\text{depends on Jacobian}} \underbrace{(x_0 - y_0)}_{\text{fixed}}$$

Therefore,

$$\to \quad \|x_n - y_n\| \approx e^{\Lambda(x_0)\, n} \cdot \|x_0 - y_0\|.$$

- If $\Lambda(x_0) > 0$, then $e^{\Lambda(x_0)} = N(x_0)$ grows and the

- If $\Lambda(x_0) > 0$, then $e^{\Lambda(x_0)} \neq V(x_0)$ grows and the orbits will separate (indicating instability).
- If $\Lambda(x_0) < 0$, then $e^{\Lambda(x_0)}$ decays and the orbits get closer (indicating stability).

Remark: In one dimension, so $X \subseteq \mathbb{R}$, then

$$\rho(Df^n(x_0)) = |f'(x_0)|\,|f'(x_1)| \cdots |f'(x_{n-1})|.$$

$$\Lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)|.$$

Ex 6.9: Let $f(x) = 4x(1-x)$, and $x_0 = \frac{1}{4}$.

First, we compute the orbit of x_0 under f.

$$\mathcal{O}^+(x_0) = \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \ldots\right)$$

($\frac{1}{4}$ is eventually periodic.) Then

$$f'(x) = 4 - 8x,$$

and $f(\frac{1}{4}) = 4 - 2 = 2$, and $f(\frac{3}{4}) = 4 - 6 = -2$.

The "spectral radius" of Df (which is just f') is just $|f'(x)|$. Thus,

$$|f'(1/4)| = |f'(3/4)| = 2.$$

By definition,

$$\begin{aligned}
\Lambda(x_0) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(2) \\
&= \lim_{n\to\infty} \frac{1}{n} \left(n \cdot \log(2)\right)
\end{aligned}$$

$$= \lim_{n\to\infty} \log(2)$$
$$= \log(2).$$

So $N(x_0) = e^{\Lambda(x_0)} = 2.$ $\square$

Suggests that the orbit $O^+(1/4)$ is unstable, and points nearby will grow farther apart.

6.2: Some remarks on chaotic behavior.

What makes a system "chaotic"?

- hard to predict
  - ↳ also includes small changes to initial conditions have unpredictable consequences.

Def: Let $f: X \to X$. The orbit $O^+(x)$ is stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\|x - y\| < \delta$ implies that for all $n \in \mathbb{N}$,

$$\|f^n(x) - f^n(y)\| < \varepsilon.$$

the orbit is unstable if it is not stable.

Stability of an orbit is determined by the distance of orbits nearby. It is stable if the orbits remain close

We can also discuss stability relative to a subset $Y \subseteq X$. That is, the points y in the above definition are then assumed to be contained in Y. (Also, $f(Y) \subseteq Y$.)

Ex: 6.11 $f(x) = x(1-x)$, and let's consider the stability of the orbit $O^+(0)$.

For all $w < 0$.

the stability of the orbit $O(0)$.

For all $y<0$,

$$f(y) = \underbrace{y}_{<0}\cdot\underbrace{(1-y)}_{>1} < y < 0.$$

Therefore, $f^n(y)<0$ for all $n\in\mathbb{N}$. In particular

$$\lim_{n\to\infty} f^n(y) = -\infty.$$

However, the orbit $O^+(0)$ is stable relative to $[0,1]=:Y$. First, we show that $f(Y)\subseteq Y$. Suppose $y\in Y$, then $f(y)=\underline{y}\,(\underline{1-y})$.

$$y\in Y \qquad 1-y\in Y$$

Thus, $f(y)\in Y$, so $f(Y)\subseteq Y$. Now, we show that for all $\varepsilon>0$, we can find $\delta>0$ s.t. $\forall y\in Y$, with $\|0-y\|<\delta$, then for all $n\in\mathbb{N}$,

$$\|f^n(0)-f^n(y)\|<\varepsilon.$$

Let $\varepsilon>0$. Choose $\delta = ???$. For $y\in Y$, with $|y|<\delta$, then $\forall n\in\mathbb{N}$

$$|\underline{f^n(0)} - f^n(y)| = |0-f^n(y)| = |f^n(y)|$$

Choose $\delta=\varepsilon$!

We want to understand how $|f^n(y)|$ is related to δ. $\quad y\in[0,1]$ and $|y|<\delta$

$$f(y) = \underline{y}\,\underline{(1-y)} \quad \begin{cases} y\in[0,1] \text{ and } |y|<\delta \\ 1-y\in[0,1]\end{cases}$$

$$|f(y)| = |y\underbrace{(1-y)}_{\leq 1}| \leq |y| < \delta$$

Thus, $|f(y)|<\delta$. If we start with y s.t. $|y|<\delta$, then $|f(y)|<\delta$. Therefore, for all $n\in\mathbb{N}$, $\underline{|f^n(y)|<\delta}$.

We showed that the quantity we want to be smaller than ε is in fact smaller than δ.

So if we choose $\delta=\varepsilon$, then for all $y\in Y$ with $|y|<\delta$, then $\forall n\in\mathbb{N}$, with $|y|<\delta$, then $\forall n\in\mathbb{N}$,

$$|f^n(y)| < |y| < \delta = \varepsilon.$$

Thus, $x=0$ is stable relative to $[0,1]$. $\square$

<u>Def</u>: $f: X\to X$. We say f has **sensitive dependence** if $\exists\, \varepsilon>0$ s.t $\forall x\in X$ and any $\delta>0$, $\exists\, y\in X$ and $n\in\mathbb{N}$ s.t $\|x-y\|<\delta$ and

$$\|f^n(x)-f^n(y)\|\geq \varepsilon.$$

No matter which two points are chosen, their orbits will never stay close forever.