Lecture 15

Friday, June 12, 2020 8:25 AM

Two kinds of seqs.

$$\Big(\underbrace{\|A^n x\|}_{\mathbb{R}}\Big)_{n=0}^{\infty}$$

$\mathbb{R}$ because $\|\cdot\|$ returns $\mathbb{R}$.

$$(A^n)_{n=0}^{\infty} = (\underset{0}{I}, \underset{1}{A}, \underset{2}{A^2}, \underset{3}{A^3}, \ldots)$$

A sequence is just a function $S: \mathbb{N}_0 \to X$.

It gets tricky when determining convergence.

When $X \subseteq \mathbb{R}$, the def is: $\forall \varepsilon > 0, \exists N > 0$ such that $\forall n \geq N$, $|x_n - \bar{x}| < \varepsilon$.

At some point in the (real-valued) sequence, all entries x_n are within ε-distance from the limit $\bar{x}$.

For $X \subseteq Mat_{d \times d}(\mathbb{R})$, we need $|x_n - \bar{x}| < \varepsilon$ to make sense. We can subtract matrices, but we need something to replace $|\cdot|$.

Use matrix norms.

$$\|A\| = \sqrt{\sum_{i,j=1}^{d} a_{ij}^2} \in \mathbb{R}.$$

Back to non-linear disc. systems.

A nonlinear system has the form

$$x_{n+1} = f(x_n),$$

where each $x_n \in \mathbb{R}^d$, and $f: X \to X$, with $X \subseteq \mathbb{R}^d$. Some quick notation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}, \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_d(x) \end{pmatrix}.$$

Recall theorems like Theorem 3.5.

$$|f'(x)| < 1 \quad \text{or} \quad |f'(x)| > 1.$$

What do we do here? The Jacobian acts like "the" derivative. The **Jacobian** is the matrix

$$\begin{array}{cccc} & x_1 & x_2 \cdots & x_d \\ f_1 & \Big(\frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_d}(x) \Big. \end{array}$$

the matrix

$$Df(x) = \begin{array}{c|c} & \begin{array}{ccc} x_1 & x_2 \cdots & x_d \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_d \end{array} & \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_d}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial x_1}(x) & \cdots & \frac{\partial f_d}{\partial x_d}(x) \end{pmatrix} \end{array}$$

Suppose $\bar{x}$ is a fixed point of the system. Thus, $\bar{x} = f(\bar{x})$. Then if we expand f via its Taylor series, this gives:

$$f(x) = f(\bar{x}) + Df(\bar{x})(x-\bar{x}) + \overbrace{\mathcal{O}(\|x-\bar{x}\|^2)}^{\text{higher order terms}}$$

$$\boxed{f(x) \approx \bar{x} + Df(\bar{x})(x-\bar{x}).}$$

This is only true <u>near</u> $\bar{x}$.

<u>Def.</u> let $f: X \to X$ with $X \subseteq \mathbb{R}^d$ and f is C^1. A fixed pt. $\bar{x}$ is **hyperbolic** if all eigenvalues λ of $Df(\bar{x})$ $|\lambda| \neq 1$. (This is the <u>same</u> def. if $d=1$.)

<u>Thm 6.2:</u> Let f be a C^1-function and $\bar{x} \in X$ a fixed pt. of f.

(i) The f.p. $\bar{x}$ is stable and attracting if $|\lambda| < 1$ for all eigenvalues of $Df(\bar{x})$.

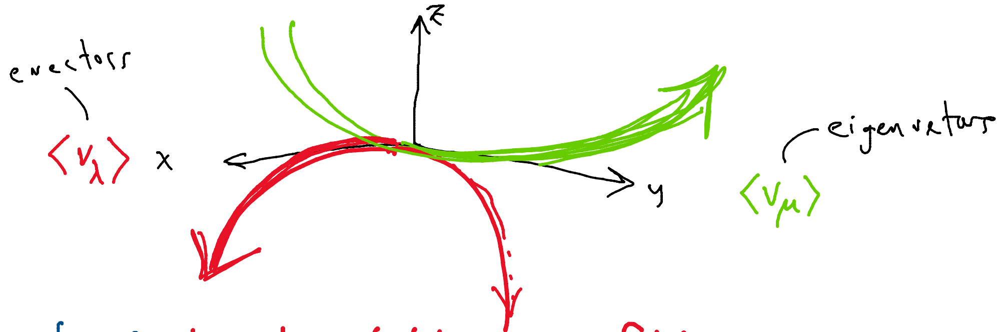
(ii) The f.p. $\bar{x}$ is unstable if $\exists$ an eval λ of $Df(\bar{x})$ s.t. $|\lambda| > 1$.

(iii) the f.p. $\bar{x}$ is unstable and repelling if $\forall$ evals λ of $Df(\bar{x})$, $|\lambda| > 1$.

When $|\lambda| = 1$, this is inconclusive: see examples from Ch 5.

<u>Def.</u> Same hypotheses as Thm 6.2. The fixed point $\bar{x}$ is a **saddle point** if $\exists$ evals λ, μ of $Df(\bar{x})$ s.t. $|\lambda| < 1 < |\mu|$.

$Df(\bar{x})$ s.t. $|\lambda| < 1 < |\mu|$.



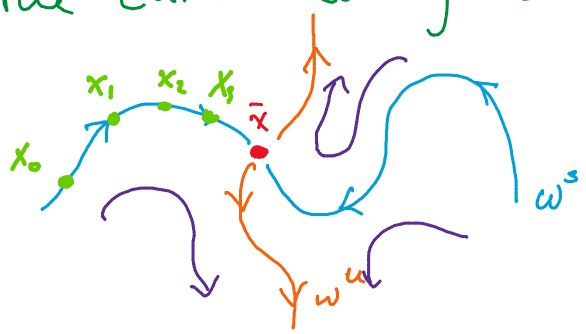
Def: A local stable manifold is a region W^s such that $\forall x \in W^s$,

$$\lim_{n\to\infty} f^n(x) = \bar{x}. \qquad \text{(forward)}$$

A local unstable manifold is a region W^u s.t. $\forall x \in W^u$,

$$\lim_{n\to\infty} f^{-n}(x) = \bar{x}. \qquad \text{(backward)}$$

Remark: W^s can be seens as the curve "going into" $\bar{x}$ and W^u can be seen as the curve coming "out of" $\bar{x}$.



Ex 6.5: Suppose

$$f(x,y) = \begin{pmatrix} \frac{x}{2} \\ 2y - \frac{15}{8}x^3 \end{pmatrix}. \quad \begin{matrix} f_1 \\ f_2 \end{matrix}$$

$$f(x,y) = \left(x/2,\ 2\underset{t^3}{y} - \frac{15}{8}\underset{t}{x^3}\right). \quad \to \frac{16}{8}t^3 - \frac{15}{8}t^3$$

Note that $(0,0)$ is a fixed point.

$x = x_1,\ y = x_2$

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\cdots}{\partial x} & \frac{\cdots_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{45}{8}x^2 & 2 \end{pmatrix}$$

At $(0,0)$: $$Df(0,0) = \begin{pmatrix} \overset{\lambda_1}{\frac{1}{2}} & 0 \\ 0 & 2 \end{pmatrix} \lambda_2$$

Since $|\frac{1}{2}| < 1 < |2|$, $(0,0)$ is a saddle point. Note that $f(0,t) = (0,2t)$, so

$$f^n(0,t) = (0, 2^n t)$$

$$f^{-n}(0,t) = \underbrace{(0, 2^{-n} t)}_{\to (0,0)}. \quad n \to \infty$$

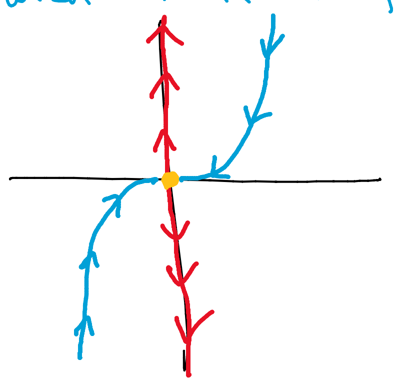
This means the points of the form $(0,t)$, for all $t \in \mathbb{R}$, form the local unstable manifold.
For the stable consider:

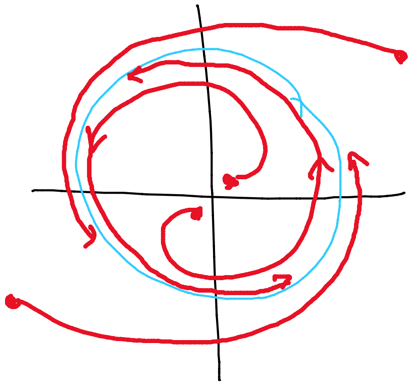
$$f(t,t^3) = \left(t/2, \, t^3/8\right) = \left(t/2, \, (t/2)^3\right).$$

This means that

$$f^n(t,t^3) = \left((t/2)^n, \, (t/2)^{3n}\right)$$

This is the local stable manifold.

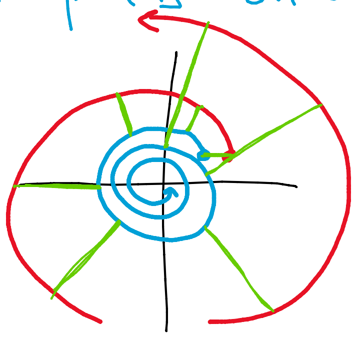




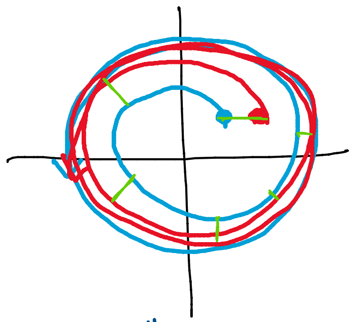
← Read about this in Remark 6.6.

6.1: Intro to Lyapunov exponents.

Informally Lyapunove exp. is a number that measures how two close paths diverge.



A large Lyapunov exp.



Small Lyapunov exp.

This is one way to "classify" whether a system is chaotic: large Lypanov exp.

This is a way to measure how sensitive the system is to changes in the initial conditions.

Def: Let $A \in Mat_{d \times d}(\mathbb{R})$ with eigenvalue $\lambda_1, \dots, \lambda_d$. The **spectral radius** of A is the number
$$\rho = \max_i |\lambda_i|.$$

From previous example: $Df(0,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$.
the spectral radius is 2.

→ 1. $\bar{x}$ is stable if $\rho(Df(\bar{x})) < 1$,
unstable if $\rho(Df(\bar{x})) > 1$.

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unstable if $\rho(Df(\bar{x})) > 1$.

2. A periodic point $\bar{x}$ of period p is stable if $\rho(Df^p(\bar{x})) < 1$ and unstable if $\rho(Df^p(\bar{x})) > 1$. This is equivalent to

$$\text{stable} \equiv \left(\rho(Df^p(\bar{x}))\right)^{1/p} < 1.$$

$$\text{unstable} \equiv \left(\rho(Df^p(\bar{x}))\right)^{1/p} > 1.$$

In general the spectral radius of $Df^n(x_0)$ is equal to

$$(*) \qquad Df^n(x_0) = Df(x_{n-1})\,Df(x_{n-2}) \cdots Df(x_1)\,Df(x_0).$$

If $\underbrace{x_0 = \bar{x}}_{\text{fixed pt.}}$, then $Df^n(x_0) = (Df(x_0))^n$, implying

that
$$\underline{\rho(Df(x_0)) = \left(\rho(Df^n(x_0))\right)^{1/n}}.$$

6.8

<u>Def.</u> Let $X \subseteq \mathbb{R}^d$ and $f: X \to X$. Let $x_0 \in X$ s.t. $\rho(Df(x_n)) > 0$ for all $n \in \mathbb{N}$. The **Lyapunov number** of the orbit $O^+(x_0)$ is, provided the limit exists,

$$N(x_0) = \lim_{n \to \infty} \left(\rho(Df^n(x_0))\right)^{1/n},$$

and the **Lypanov exp.** of $O^+(x_0)$ is

$$\Lambda(x_0) = \log N(x_0) = \lim_{n \to \infty} \frac{1}{n} \log \rho(Df^n(x_0)).$$

Eigenvalues of general $A \in Mat_{d \times d}(\mathbb{R})$

$$Ax = \lambda x.$$

Eigenvalues of general $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$.

$$A\underline{x} = \lambda \underline{x}.$$

$$A\underline{x} - \lambda \underline{x} = \underline{0}$$

$$(\underbrace{A - \lambda I_d}_{\text{mat}})\, x = 0 \qquad I_d = \overbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}^{d} \Big]\, d$$

Now, we solve for eigenvalues/vectors by solving:

$$\underline{(A - \lambda I)\, x = 0.}$$

This means x is in the nullspace or kernel of the matrix $A - \lambda I$.

1. Eigenvectors come from a basis of the nullspace of $A - \lambda I$.
2. Eigenvalues come from finding the roots of the polynomial given by

characteristic polynomial $\rightarrow$ $p_A(\lambda) := \det(A - \lambda I)$.

$$\underline{p_A(\lambda) = 0} \leftarrow \text{characteristic equation.}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 2 \\ -2 & 3-\lambda \end{pmatrix}.$$

Eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(3-\lambda) - (-2)(2) \\ &= (3 - 4\lambda + \lambda^2) + 4 \\ &= \lambda^2 - 4\lambda + 7. \end{aligned}$$

The solutions are:

$$\begin{aligned} \lambda &= \frac{4 \pm \sqrt{4^2 - 4(1)(7)}}{2} \\ &= 2 \pm \tfrac{1}{2}\sqrt{16 - 28} \\ &= 2 \pm \tfrac{1}{2}\sqrt{-12} \\ &= 2 \pm \sqrt{-3} \end{aligned}$$

... the nullspace.

$= 2 \pm \sqrt{-3}$

$$\begin{pmatrix} 1-(2+\sqrt{-3}) & 2 \\ -2 & 3-(2+\sqrt{-3}) \end{pmatrix} \longleftarrow \text{Find the nullspace.}$$