When $|f_a(0)| = 1$, can apply Prop 3.8.

$\hookrightarrow x_0 \neq 0$: translate system.

Say $|f_a(x_0)| = 1$, but $x_0 \neq 0$

Define $\boxed{g_a(x) = f_a(x - x_0)}$

So $x = 0$ is fixed pt of g_a.

eigen-values

Prop. 5.11. If $A \in Mat_{d\times d}(\mathbb{R})$ with all evals $|\lambda| < 1$, then $x = 0$ is stable and attracting.

Proof idea:

Use thm 5.10, which states that

$$\lim_{n\to\infty} A^n = 0$$

We know that

$$\lim_{n\to\infty} A^n x_0 = 0 x_0 = 0.$$

$x = 0$ is attracting.

To show stable requires a little more work. The sequence

$$(\|A^n\|)_{n=1}^{\infty}$$

is bounded. Use this fact, and show that

$$\|x_n\| \leq \underbrace{\boxed{\alpha}}_{\text{some bound}} \|x_0\|.$$

Then write down $\delta = \varepsilon/\alpha$ so that if

$$\|x_0\| < \delta, \text{ then } \underbrace{\|x_n\| < \varepsilon}_{\text{within } \varepsilon \text{ of } 0}.$$

Thm 5.12 If $A \in Mat$ ... $(\mathbb{R})$ ...

within ε of 0.

Thm 5.12. If $A \in Mat_{d\times d}(\mathbb{R})$ has an eval λ s.t. $|\lambda| > 1$, then $\exists\, x \in \mathbb{R}^d$ s.t.

$$\lim_{n\to\infty} \|A^n x\| = \infty.$$

Cor. 5.13 Same hypothesis at Thm 5.12, then $x = 0$ is unstable.

Cor. 5.14 If $A \in Mat_{d\times d}(\mathbb{R})$ where every eval λ satisfies $|\lambda| > 1$, then $x = 0$ is unstable and repelling.

Proof of Thm 5.12:

If $\lambda \in \mathbb{R}$, then $\exists\, x \in \mathbb{R}$ s.t. $Ax = \lambda x$. Then

$$\|A^n x\| = \|A^{n-1} \cdot Ax\|$$
$$= \|\lambda^n x\| = |\lambda|^n \|x\|.$$

Since $|\lambda| > 1$, it follows that $\|A^n x\| \to \infty$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then the same idea holds, but it is a little complicate.

Then its eigenvector $w = x + iy$ where both x and y are L.I. — Linear independent

From these x, y, we get

$$A^n w = A^n x + i A^n y = r^n \big(x\cos(n\theta) - y\sin(n\theta)\big)$$
$$+ i r^n \big(x\sin(n\theta) + y\cos(n\theta)\big)$$

where $\lambda = re^{i\theta}$.

Compare real and imaginary parts.

$z = a + bi$: $Re(z) = a$
$Im(z) = b$

$\mathrm{Im}(z) = b$

$$A^n x = r^n(x\cos(n\theta) - y\sin(n\theta))$$

Oj

$$A^n y = \text{similar.}$$

Now for the norm

$$\|A^n x\|^2 \geq \underline{r^{2n}(\text{ complicated })}.$$

Goal show that the right side goes to ∞ as $n \to \infty$. To show this, we prove that $\underline{(\text{complicated}) > 0}$.

Since $|\lambda| = r > 1$, $r^{2n} \to \infty$ as $n \to \infty$, and if $(\text{complicated}) > 0$, then the whole quantity $\to \infty$ as $n \to \infty$.

$$(\text{complicated}) = \underline{(\cos\phi)^2\|x\|^2 + (\sin\phi)^2\|y\|^2} \quad \phi = n\theta.$$
$$\boxed{-2|\cos\phi||\sin\phi||\langle x, y\rangle|}$$

Show is greater than 0 by Cauchy-Schwartz inequality:

$$\|x\|\|y\| > |\langle x, y\rangle|.$$

This shows $(\text{complicated}) > 0$. $\square$

Thm 5.16: If $A \in \mathrm{Mat}_{d \times d}(\mathbb{R})$ with evals s.t. $|\lambda| \notin \{0, 1\}$, then $\exists$ subspaces V_s and V_u of $\mathbb{R}^d$ with $\dim V_s + \dim V_u = d$ satisfying:

1. If $x \in V_s$, then $\lim_{n\to\infty} A^n x = 0$,
2. If $x \in V_u$, then $\lim_{n\to\infty} A^{-n} x = 0$,
3. If $x \notin V_s$, then $\lim_{n\to\infty} \|A^n x\| = \infty$.

Remarks: If A has no 0 evals, then A is invertible. The subspace V_s is called the stable subspace and V_u the unstable subspace.

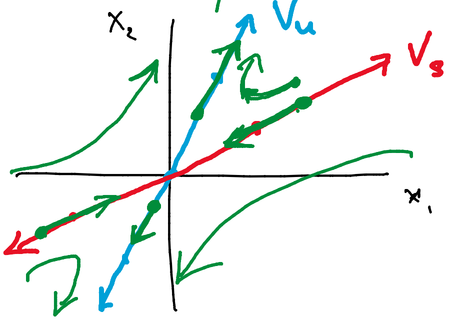
Ex: 5.17 Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 5/2 \end{pmatrix}.$$

The evals of A are $\boxed{1/2}$ and $\boxed{2}$, and the evecs. are

$$\boxed{u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}, \quad \boxed{v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

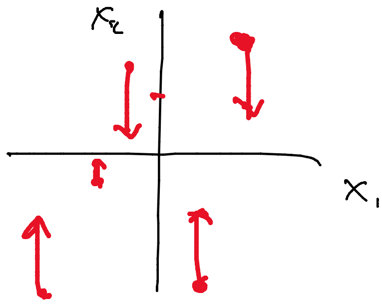
These form a basis of $\mathbb{R}^2$ and u spans V_s and v spans V_u.



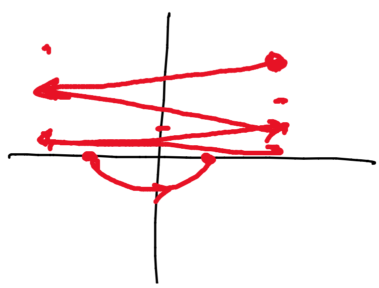
Thus $x=0$ is unstable and is not repelling.

Ex 5.18:

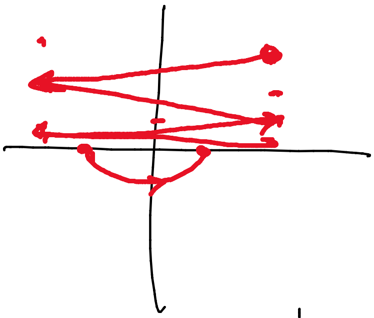
1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$



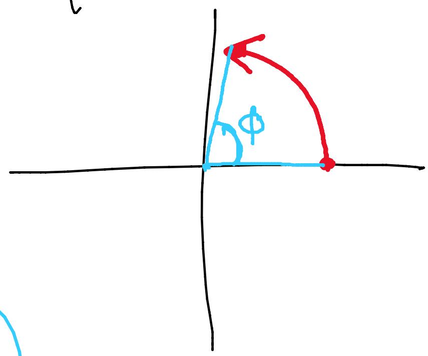
2. $A = \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}$



2. $A = \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}$

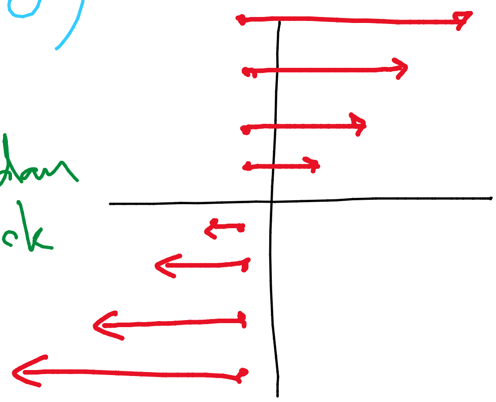


3. $A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$



$\phi = \frac{\pi}{2}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

4. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ = Jordan block



shear transform

Now, you should be able to determine the stability of $x=0$ in the system $x_{n+1} = A x_n$ based on evals of A.

Ch6: Nonlinear discrete dynamical sys.

These systems are, in general, hard to analyze. We need a good balance between [illegible] models.

to analyze. We need a good balan[illegible]

simple and complicated models.

Logistic model is non-linear.

Models that are not continuous are always non linear (because linear models are continuous always).

$$x_{n+1} = f(x_n),$$

where $f: X \to X$, with $X \subseteq \mathbb{R}^d$.

An example: $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n^2 - y_n^2 \\ 2x_n y_n \end{pmatrix}$

In this example $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}.$$