Lecture 13

Friday, June 5, 2020 8:27 AM

Find periodic points of period $m \in \mathbb{N}$ where

$$f(x) = 2|x| - 1.$$

$$f(x) = \begin{cases} 2x - 1 & x \geq 0, \\ -2x - 1 & x < 0. \end{cases}$$

$$f^2(x) = f(2|x| - 1)$$

$$= \begin{cases} f(2x - 1) & x \geq 0, \\ f(-2x - 1) & x < 0. \end{cases}$$

$$= \begin{cases} 2|2x - 1| - 1 & x \geq 0 \\ 2|2x + 1| - 1 & x < 0 \end{cases}$$

$-1/2 \leq x < 0$

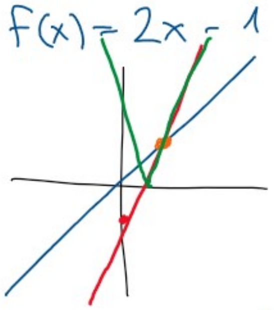
$x < -1/2$

You can write this explicitly. It's not easy.

Apply Sharkovsky's Thm.

Only need $m = 3$.

Think $f(x) = 2x - 1$



No periodic points (not fixed).

$f(2|x| - 1)$

Fix $x_0 \in \mathbb{R}$. $x_1 = 2|x_0| - 1$

Guess that $x_0 < 0$, $x_1 = -2x_0 - 1$.

Say that $x_1 \geq 0$, so

$$x_2 = f(x_1) = 2x_1 - 1$$
$$= -4x_0 - 2 - 1$$
$$x_2 = -4x_0 - 3.$$

Need $x_3 = x_0$.

$\cancel{x_0, x_1, x_2}$.

Consider $f(x) = x^2 - x + 1$.

$f^3(x)$: is a poly of deg 6.

Ch 5.

Section 5.1: Matrix Norms.

Throughout we have fixed basis $e_1, \dots, e_d \in \mathbb{R}^d$, so every vector

$$x = \sum_{i=1}^{d} x_i e_i = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$

The dot product gives a norm.

$$\langle x y \rangle = x \cdot y = \sum_{i=1}^{d} x_i y_i = x_1 y_1 + \cdots + x_d y_d.$$

This gives rise to the L^2-norm.

$$\|x\| := \|x\|_2 = \langle x, x \rangle^{1/2} = \sqrt{\sum_{i=1}^{d} x_i^2}.$$

We can view matrices a vectors: they can be added, scaled, There is a basis.

Def. A matrix norm $\|\cdot\|$ on $Mat_{d\times d}(\mathbb{R})$ is

$$\boxed{\|A\| = \sqrt{\sum_{i,j=1}^{d} |a_{ij}|^2}}.$$

Prop 5.3: If $A \in Mat_{d\times d}(\mathbb{R})$, then $\forall x \in \mathbb{R}^d$,

$$\|Ax\| \leq \|A\| \cdot \|x\|.$$

Proof: For $i \in \{1, \dots, d\}$ let A_i be the i^{th} row of A. Then

$$Ax = \begin{pmatrix} \langle A_1, x \rangle \\ \langle A_2, x \rangle \\ \vdots \\ \langle A_d, x \rangle \end{pmatrix} \leftarrow$$

$$Ax = \begin{pmatrix} \langle A_1, x\rangle \\ \vdots \\ \langle A_d, x\rangle \end{pmatrix}.$$

Therefore,

$$\|Ax\|^2 = \sum_{i=1}^{d} \langle A_i, x\rangle^2 \qquad \langle A_i, x\rangle^2 \leq \langle A_i, A_i\rangle \cdot \langle x, x\rangle$$

$$\leq \sum_{i=1}^{d} \left(\langle A_i, A_i\rangle \cdot \langle x, x\rangle \right)$$

$$= \langle x, x\rangle \sum_{i=1}^{d} \langle A_i, A_i\rangle$$

$$= \langle x, x\rangle \sum_{i=1}^{d} \underbrace{\boxed{\sum_{j=1}^{d} (A_i)_j (A_i)_j}}_{\text{def } \bullet}$$

$$= \langle x, x\rangle \sum_{i,j=1}^{d} A_{ij}^2$$

$$= \underbrace{\langle x, x\rangle}\, \|A\|^2$$

$$= \|x\|^2 \|A\|^2. \qquad \square$$

5.2: Jordan Blocks

Every square matrix can be expressed, under the right basis, with Jordan Blocks.

Def: A square matrix J is a Jordan block if it is of the form:

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \pi & 1 & & \\ & \pi & 1 & \\ & & \pi & 1 \\ & & & \pi \end{pmatrix}$$

Def: We say a square matrix A is in Jordan Normal Form if

$$\begin{pmatrix} J_1 & & 0 \\ & J & \end{pmatrix} \qquad J_1 \quad \left(\begin{array}{ccc|ccc} 2 & 1 & & & & \\ & 2 & 1 & & 0 & \\ & & 2 & & & \\ \hline & & & \pi & 1 & \\ & & & & \pi & 1 \\ 0 & & & & & \end{array}\right)$$

$$A = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix}, \qquad \begin{pmatrix} \ddots & 2 & & & & \\ & & \pi & 1 & & \\ & & & \pi & 1 & \\ & 0 & & & \pi & 1 \\ & & & & & \pi \end{pmatrix}_{J_2}$$

where each J_i is a Jordan Block

<u>Thm</u> (Jordan Normal Form). For every square matrix A, $\exists$ an invertible matrix S s.t. $\underbrace{S^{-1}AS}_{\text{change of basis.}}$ is in JNF.

<u>Ex:</u> 5.6

$$A = \begin{pmatrix} 2 & 4 & -6 & 0 \\ 4 & 6 & -3 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & -6 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1/4 & 0 & 1 \\ 0 & 1/4 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

generalized eigenvector.

$$J = S^{-1}AS = \begin{array}{c} \begin{pmatrix} 2 & 1 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & \cdot & 6 \end{pmatrix} \\ \begin{matrix} 1 & - & 2 & 3 \end{matrix} \end{array} \leftarrow \text{eigenvalues of } A$$

"there $\cdot = 0$."

There is one problem: some eigenvalues are not real.

<u>Definition:</u> A square matrix J is a real Jordan block if it is either a Jordan block as above where $\lambda \in \mathbb{R}$, or J has the form

$$\begin{pmatrix} B & I & & & \\ & B & I & & \\ & & B & \ddots & \\ & & & \ddots & I \\ & & & & B \end{pmatrix},$$

where $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $(a, b \in \mathbb{R})$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We say A is in real Jordan Normal Form if

We say A is in real Jordan Normal Form if all blocks are real Jordan blocks.

Recall, $\underline{x_n = A x_{n-1}}$. $\longrightarrow$ $x_n = A^n x_0$ where x_0 is some initial vector. Powers of A predict what will happen.

$\mathbb{R}$: reals $\mathbb{N}$: natural
$\mathbb{C}$: complex.

Lemma. Let $J \in Mat_{d\times d}(\mathbb{C})$ be a Jordan block with eigenvalue λ. If $|\lambda| < 1$, then the sequence $(J^n)_{n=0}^{\infty}$ converges to the zero matrix. If $|\lambda| > 1$, then the same seq. diverges.

matrix norm

proof: See lecture notes pdf.
Reminder: Check out 1.1 of complex numb.

What can happen when $\lambda = 1$?

$$J = (1) \qquad J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$J^n = (1) \qquad J^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

converges $\qquad$ diverges.

Thm: If A is a square matrix ~~with~~ where every eigenvalue λ satisfies $|\lambda| < 1$, then

$$\lim_{n\to\infty} A^n = \mathbb{O}.$$

zero matrix.

proof: Use JNF-theorem to write A as

$$J = S^{-1}AS$$

$$\boxed{A = SJS^{-1}}$$

Now, for $n \geq 0$

$$A^n = \underbrace{(SJS^{-1})(SJS^{-1})\cdots(SJS^{-1})}_{n \text{ times}}$$

$$= SJ^nS^{-1}$$

Therefore,

Therefore,

$$\boxed{\lim_{n\to\infty} A^n = 0 \iff \lim_{n\to\infty} J^n = 0.}$$

We know

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix}.$$

$$J_1 = \begin{pmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{pmatrix} \quad J_2 = \begin{pmatrix} \lambda_2 & 1 & & 0 \\ & \lambda_2 & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_2 \end{pmatrix}$$

Therefore,

$$J^n = \begin{pmatrix} J_1^n & & & 0 \\ & J_2^n & & \\ & & \ddots & \\ 0 & & & J_m^n \end{pmatrix}$$

Try it yourself.

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)^n.$$

By assumption every λ satisfies $|\lambda| < 1$.
By the above lemma, each Jordan block converges to the 0 matrix. So

$$\lim_{n\to\infty} J^n = \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} = 0.$$

Thus, $\lim_{n\to\infty} A^n = 0$ by above. $\square$

5.3: Stability Criteria.

Prop 5.11: If $A \in Mat_{d\times d}(\mathbb{R})$ with all eigenvalues λ with $|\lambda| < 1$, then $x=0$ is stable and attracting in the system given by $x_n = A x_{n-1}$.

proof: Read this, and ask questions on Tuesday if you have any. (p.53).