We discussed 4 types of bifurcations:

1. Saddle-node
2. Transcritical
3. Pitchfork
4. Period doubling.

These are fundamental. The examples we will look at are "built" from these.

4.5: Misc. Bifurcations.

Ex 4.11 (Generalized saddle-node)

$$f_a(x) = x - (x^2 - a)(x^2 - 4a).$$

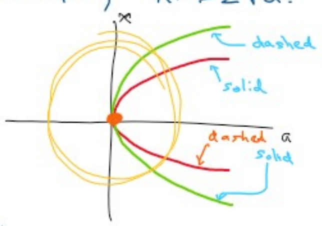
Fixed pts satisfy:

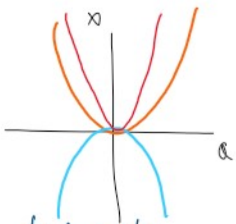
$$\boxed{x = x - (x^2 - a)(x^2 - 4a)}$$

$$\to 0 = (x^2 - a)(x^2 - 4a)$$

So we have 4 fixed points:

$$x = \pm\sqrt{a}, \quad x = \pm 2\sqrt{a}.$$





fixed points

$$0 = a^2 + kx$$

$$x = a^2 + x + x$$

$$x = a^2 + 2x$$

$$\boxed{f_a(x) = a^2 + kx}$$

We consider a few cases.

Case 1: $x = \sqrt{a}$. $f_a'(x) = 1 - \left(2x(x^2 - 4a) + (x^2 - a)(2x)\right)$

$$f_a'(x) = 1 - 2x^3 + 8ax - 2x^3 + 2ax$$

$$= 1 - 4x^3 + 10ax$$

$$\boxed{f_a'(x) = 1 - 2x(2x^2 - 5a)}$$

$a \geq 0$

$$f_a'(\sqrt{a}) = 1 - 2\sqrt{a}(2a - 5a)$$

$$= 1 + 6a\sqrt{a}$$

$$\boxed{f_a'(\sqrt{a}) = 1 + 6a^{3/2}}$$

For $a > 0$, $|f_a'(\sqrt{a})| = |1 + 6a^{3/2}| > 1$

This is always unstable and repelling for all $a > 0$.

Case 2: $x = -\sqrt{a}$.

$$f_a'(x) = 1 - 2x(2x^2 - 5a)$$

$$f_a'(-\sqrt{a}) = 1 + 2\sqrt{a}(-3a)$$

$$= 1 - 6a^{3/2}$$

Using same analysis,

$$|f_a'(-\sqrt{a})| = |1 - 6a^{3/2}|$$

For values of a close to $a = 0$,

$$|1 - 6a^{3/2}| < 1.$$

For values of a close to $a=0$,

$$|1-6a^{3/2}| < 1.$$

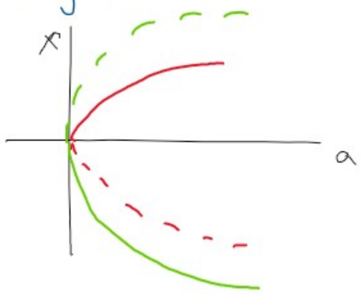
Thus, this is stable (close to $a=0$).

<u>Case 3:</u> $x = 2\sqrt{a}$

$$f_a'(2\sqrt{a}) = 1 - 4\sqrt{a}\,(3a) = 1 - 12a^{3/2}$$

This is the same kind of analysis as Case 2.

<u>Case 4:</u> Try it! :)



<u>Ex 4.12</u> (Gen. Pitchfork)

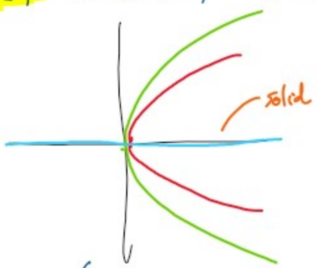
$$f_a(x) = x - x(x^2-a)(x^2-4a)$$

Fixed points: $x = f_a(x)$

$$0 = x(x^2-a)(x^2-4a).$$

5 fixed points.

$$x=0, \quad x = \pm\sqrt{a}, \quad x = \pm 2\sqrt{a}.$$



$$f_a'(x) = 1 - \Big(1\cdot(x^2-a)(x^2-4a) + x(2x)(x^2-4a) + x(x^2-a)(2x)\Big)$$

$$= 1 - \left(x^4 - 5ax^2 + 4a^2 + 2x^4 - 8ax^2 + 2x^4 - 2ax^2\right)$$

$$= 1 - \left(5x^4 - 15ax^2 + 4a^2\right)$$

$$= \boxed{1 - 4a^2 + 15ax^2 - 5x^4}$$

<u>Case 1: $x=0$</u> $\quad f_a'(0) = |1-4a^2|$

In a small neighborhood around $a=0$,

$$|1-4a^2| < 1.$$

Thus, this is stable

$$|1-4a^2| < 1.$$

Thus this is stable.

Case 2: $x = \pm\sqrt{a}$.

$$f_a'(\pm\sqrt{a}) = 1 - 4a^2 + 15a^2 - 5a^2$$
$$= 1 + 6a^2$$

~~This analysis is same as Case 1:~~

$$|f_a'(\pm\sqrt{a})| = |1+6a^2|$$

Wrong!

$$|f_a'(\pm\sqrt{a})| = |1+6a^2| > 1.$$

Thus, these fixed points are always unstable.

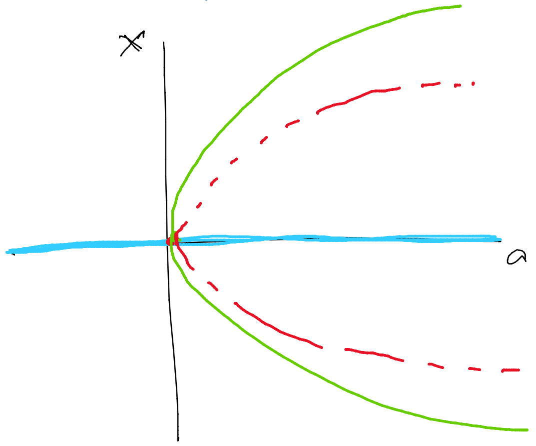
Case 3: $x = \pm 2\sqrt{a}$

$$f_a'(\pm 2\sqrt{a}) = 1 - 4a^2 + 15a(4a) - 5(16a^2)$$
$$= 1 - 4a^2 + 60a^2 - 80a^2$$
$$= 1 - 24a^2$$

This analysis is like Case 1:

$$|f_a'(\pm 2\sqrt{a})| = |1-24a^2| < 1$$

in a small neighborhood around $a=0$.



Chap 5: Linear discrete dynamical systems.

Let A be a $d\times d$ matrix: $A \in Mat_{d\times d}(\mathbb{R})$.

Define $f: \mathbb{R}^d \to \mathbb{R}^d$ by

$$f(x) = Ax.$$

Define a disc. dyn. sys. via $x_0 \in \mathbb{R}^d$

$$\boxed{x_{n+1} = f(x_n) = Ax_n}$$

In other words,

In other words,

$$\boxed{x_n = A^n x_0}$$

Let's consider fixed points:

$$x_{n+1} = A x_n$$

zero vector.

- One obvious choice is $x = 0$.
- Otherwise, we have

$$x = Ax$$

In this, x is an eigenvector of A with corresponding eigenvalue 1.

Eigen vectors/values:

A nonzero vector x is an eigenvector of A if $\exists \lambda \in \mathbb{R}$ such that

$$Ax = \lambda x.$$

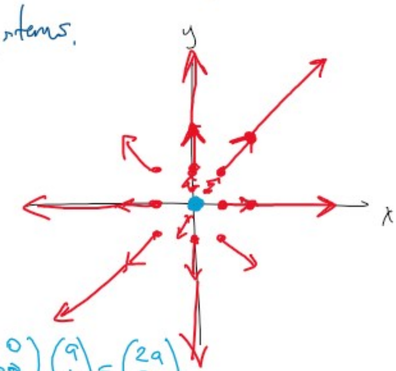
(A only scales the vector x.)

Ex: We look at various examples of linear dynamical systems.

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

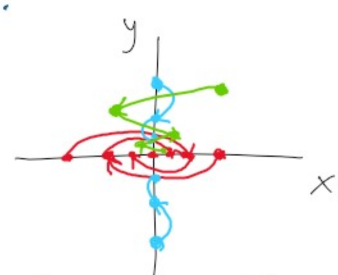
$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$

There is only 1 fixed point $x=0$. It seems that all paths go away the origin, which means it might be unstable and repelling.

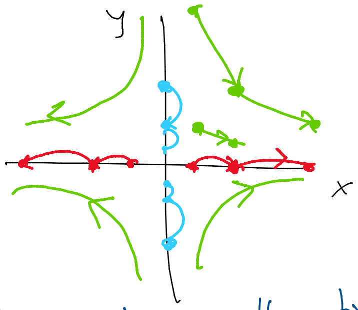
(2) $A = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$



All paths seem to go to $x=0$. This

All paths seem to go to $x=0$. This suggests that $x=0$ is stable and attracting.

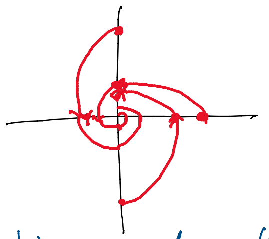
(3) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$



Seems unstable: neither attracting nor repelling.

(4) $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}}_{\text{scalar}} \cdot \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{rotation}}$

This is a rotation matrix



Seems stable and attracting.