Lecture 11

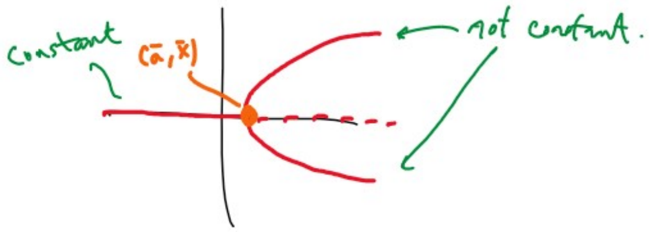
Friday, 29 May, 2020 08:13

Last time we discussed pitchfork bifurcations.

$$f(\underline{t}) = e^{\lambda \underline{t}}.$$

λ ← bifurcation parameter. → $f_a(x)$.

Pitchfork bifurcation:



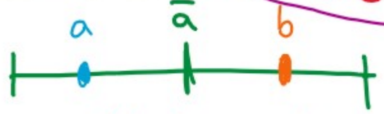
From proof of Thm 4.8: If both same sign $h''<0$

$$h''(x) = \frac{-1}{3} \boxed{\frac{F_{xxx}(\bar{a},\bar{x})}{F_{ax}(\bar{a},\bar{x})}} \neq 0.$$

Analyze key cases with pitchfork bifurcation.

Case 1: $x = \bar{x}$. Via Taylor expansion:

$$\boxed{\frac{\partial F}{\partial x}(a, \bar{x}) \approx 1 + \underbrace{\frac{\partial^2 F}{\partial x \partial a}(\bar{a}, \bar{x})}_{\neq 0} \overbrace{\underbrace{(a - \bar{a})}_{<0}}^{>0}.}$$

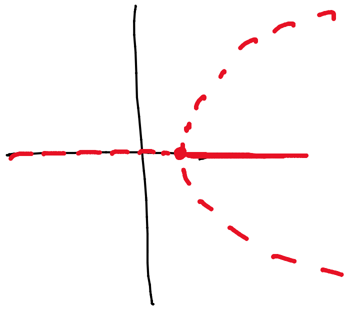


have a stability change at $a = \bar{a}$.

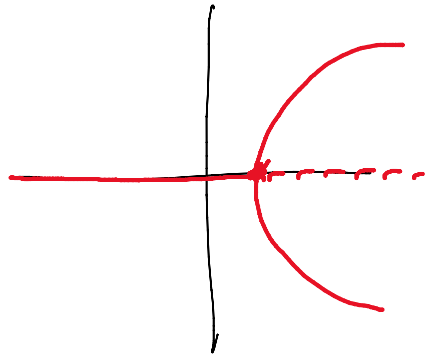
Case 2: $a = h(x)$. Via Taylor expansion

$$\frac{\partial F}{\partial x}(h(x), x) \approx 1 + \frac{1}{3} \frac{\partial^3 F}{\partial x^3}(\bar{a}, \bar{x}) \underbrace{(x - \bar{x})^2}_{\geq 0}$$

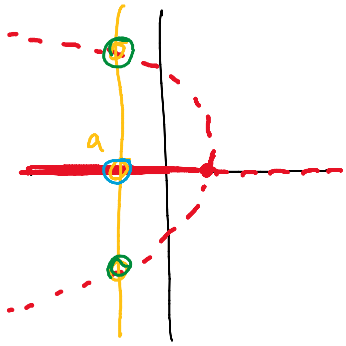
This factor $(x - \bar{x})^2$ shows that the stability of these two fixed points are the same.



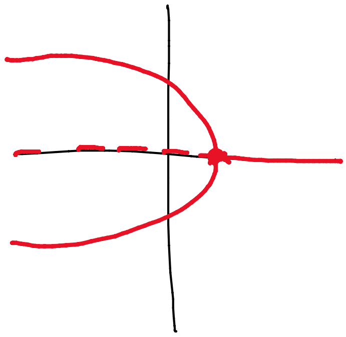
$h''>0,\ F_{xxx}>0,$
$F_{ax}<0$



$h''>0,\ F_{xxx}<0,$
$F_{ax}>0$



$h''<0,\ F_{xxx}>0$
$F_{ax}>0$



$h''<0,\ F_{xxx}<0$
$F_{ax}<0.$

This is one of many styles/kinds of bifurcations where one has a constant fixed point together with some curve $a=h(x)$..

4.4: Period Doubling Bifurcations.

This is a bifurcation that doubles the period of periodic fixed points.

Ex 4.9: $f_a(x)=ax(1-x)$. Recall we have two fixed points

- $x=0$
- $x=\frac{a-1}{\quad}$.

- $x = 0$
- $x = \frac{a-1}{a}$.

We have looked at this system for various a: $a \in (0,3)$, $a = 4$. Here, we consider $\boxed{a=3}$.

$$\boxed{x = 0,\ 2/3}$$

One way to find ~~fixed~~ periodic points is to solve

$$\boxed{f_a^p(x) = x.}$$

$$\begin{aligned} f_a^2(x) &= f_a(ax(1-x)) \\ &= a(ax(1-x))(1-(ax(1-x))) \\ &\ \vdots \\ &= -a^3x^4 + 2a^3x^3 - a^3x^2 - a^2x^2 + a^2x. \end{aligned}$$

Solve $x = f_a^2(x)$, and we get the following:

$$0 = \underbrace{x}_{\text{fixed pt}}\underbrace{\left(x - \frac{a-1}{a}\right)}_{\text{fixed pt}}\underbrace{\left(x^2 - \left(\frac{a+1}{a}\right)x + \frac{a+1}{a}\right)}_{\text{periodic pts } p=2.}$$

Sharkovsky's Thm.

$\ldots\ldots < \underline{2} < 1$

"The easiest periodic point to get is one of period 2."

Solving the quadratic:

$$x = \frac{a+1 \pm \sqrt{(a+1)(a-3)}}{2a}.$$

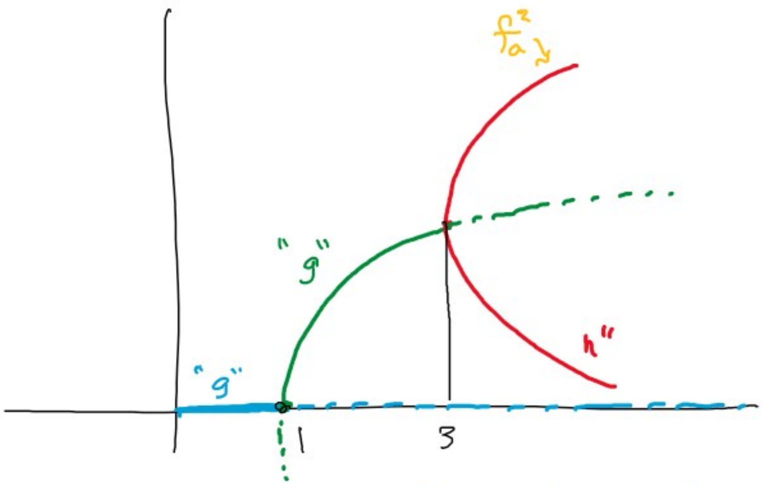
When $a < 3$, no periodic points.

When $a \geq 3$, we have periodic points.

$\underline{a = 3:}$

$$x = \frac{4 \pm \sqrt{0}}{6} = \frac{2}{3}.$$

This is the fixed point......



This is the bifurcation diagram of the logistics map near $a=3$. (Including periodic points).

Thm 4.10 (Period doubling)

Let $F : J \times I \to \mathbb{R}$ be a C^3-function. and define $F^2(a,x) := F(a, F(a,x))$. Suppose that for $(\bar{a},\bar{x}) \in J \times I$, the following hold.

1. $F(\bar{a},\bar{x}) = \bar{x}$,
2. $\dfrac{\partial F}{\partial x}(\bar{a},\bar{x}) = -1$,
3. $\dfrac{\partial^2 F}{\partial x^2}(\bar{a},\bar{x}) = 0$,
4. $\dfrac{\partial F^2}{\partial a}(\bar{a},\bar{x}) = 0$,
5. $\dfrac{\partial^2 F^2}{\partial a \partial x}(\bar{a},\bar{x}) \neq 0$,
6. $\dfrac{\partial^3 F^2}{\partial x^3}(\bar{a},\bar{x}) \neq 0$.

Then $\exists\, U \subseteq J$ and a unique C^3-function $g : U \to I$ s.t. $g(\bar{a}) = \bar{x}$, $F(a, g(a)) = g(a)$ for all $a \in U$. In addition, $\exists\, V \subseteq I$ and a unique C^2-function $h : V \to I$ s.t. $h(\bar{x}) = \bar{a}$, $F^2(h(x), x) = x$, $h'(\bar{x}) = 0$, $h''(\bar{x}) \neq 0$.

Here, g is the function IFT dealing with fixed points, where h is dealing with periodic points of period 2.

Small analysis.

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Note:

$$\frac{\partial F^2}{\partial x}(a,x) = \frac{\partial}{\partial x}\Big(F\big(a,F(a,x)\big)\Big)$$

$$= \frac{\partial F}{\partial x}\big(a, \underline{F(a,x)}\big)\frac{\partial F}{\partial x}(a,\underline{x}).$$

Since $F(a,g(a)) = g(a) = x$,

$$\frac{\partial F^2}{\partial x}(a,g(a)) = \left(\frac{\partial F}{\partial x}(a,g(a))\right)^2.$$

Since $\frac{\partial F}{\partial x}(\bar{a},\bar{x}) = -1$, then $\frac{\partial F^2}{\partial x}(\bar{a},\bar{x}) = 1$.

Also by assumptions,

$$\frac{\partial^2 F^2}{\partial a\,\partial x}(\bar{a},\bar{x}) \neq 0.$$

Thus, at $(\bar{a},\bar{x})$, the stability changes by Thm. 3.5.