Lecture 10

Tuesday, May 26, 2020 2:12 PM

Contraction: A function $f: X \to X$, where $X \subseteq \mathbb{R}$ is a contraction if $\exists K \in [0,1)$ such that for all $x, y \in X$,

$$\boxed{\|f(x) - f(y)\| \leq K\|x - y\|.}$$

So K is chosen first, which applies for all pairs $(x,y) \in X^2$.

Ex: $f(x) = 5$.

This is a contraction with $K=0$.

Let $x, y \in \mathbb{R}$

$$\|f(x) - f(y)\| = \|5 - 5\| = 0 \leq K\|x - y\|.$$

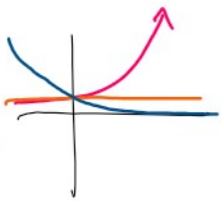
So f is a contraction.

4.1 & 4.2 are complicated.... What are we really doing?

We are looking at families of functions

$$f_a(x)$$

$$f_\lambda(t) = e^{\lambda t}.$$



f_{-16}, f_{π^2}, $f_{1,000}$ f_λ

This tells us how the long term behavior of f_a depends on a. We are looking $F(a, x)$.

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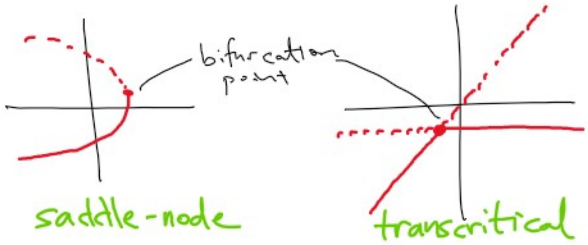
1. Saddle-node,
2. Transcritical.



4.3 Pitchfork Bifurcation.

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Ex: $f_a(x) = x + ax - x^3$.

Fixed points must satisfy:

$$x = x + ax - x^3$$

$$\boxed{0 = ax - x^3}$$

$$0 = x(a - x^2).$$

We have 3 fixed points:

$x = 0$, $x = \pm\sqrt{a}$ → $x^2 = a$

This is a pitchfork bifurcation.

Ψ

Let's understand stability.

Case 1: $a < 0$. $f_a'(x) = a + 1 - 3x^2$

Thm 3.5 determines stability based on

$$|f_a'(0)| = |a+1|$$

For $a < 0$ but close to 0, the fixed pt. $x = 0$ is stable.

Case 2: $a = 0$. $f_0'(x) = 1 - 3x^2$.

Here, $x=0$ is not hyperbolic, that means that

$$|f_0'(0)| = |1| = 1.$$

So, here, $f_0(x) = x - x^3$.

$$|f_0(x)| = |x - x^3| = |x|\,|1 - x^2|$$

Look at neighborhood around $x=0$, say $|x| < 1$. Thus, $|1 - x^2| \leq 1$.

(Prop 3.8: If $\exists r > 0$ s.t. $\forall x \in B_r(0)\setminus\{0\}$ where $|f(x)| < |x|$, then $x=0$ is an attracting fixed point.)

If $x \in B_1(0)\setminus\{0\}$, then $|1 - x^2| < 1$.

$$|f_0(x)| = |x| \cdot |1 - x^2| < |x|.$$

So by Prop 3.8, $x=0$ is stable. <1

<u>Case 3:</u> $a>0$. The deriv. is

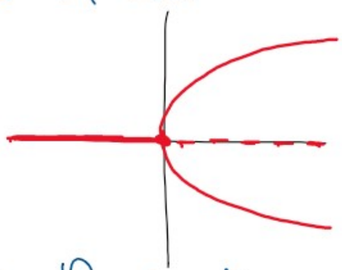
$$f_a'(x) = 1+a-3x^2.$$

Thm 3.5 says since $|f_a'(0)| = |1+a| > 1$, then $x=0$ is unstable.

Since $a>0$, we have $x=\pm\sqrt{a}$ as fixed points as well.

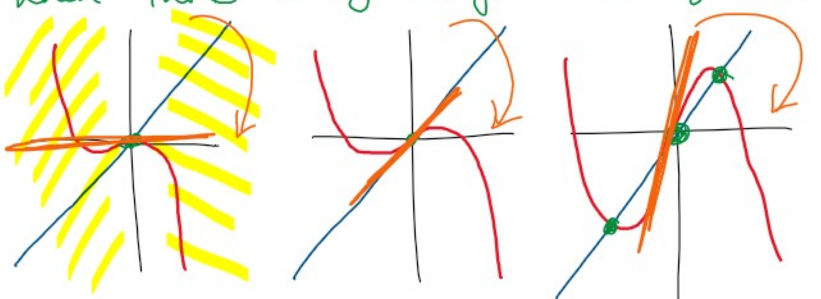
same stability
$$\left\{\begin{array}{l} |f_a'(\sqrt{a})| = |1+a-3a| = |1-2a| < 1 \\ |f_a'(-\sqrt{a})| = \cdots\cdots = |1-2a| < 1 \end{array}\right.$$

For small positive a, these fixed points are stable.



This is the b. diagram from our example.

The pitchfork bifurcation arises when there is symmetry in the system.



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<u>Thm 4.8</u> (Pitchfork bifurcation special case)

Let $F: J\times I \to \mathbb{R}$ be a C^3-function.

Suppose $\exists (\bar{a},\bar{x}) \in J\times I$ s.t.

1. $F(a,\bar{x}) = \bar{x}$ for all $a\in J$,
2. $\frac{\partial F}{\partial x}(\bar{a},\bar{x}) = 1$,

3. $\frac{\partial^2 F}{\partial x^2}(\bar{a}, \bar{x}) = 0$ (this was $\neq 0$ in transcritical.)

4. $\frac{\partial^2 F}{\partial x \partial a}(\bar{a}, \bar{x}) \neq 0$, and

5. $\frac{\partial^3 F}{\partial x^3}(\bar{a}, \bar{x}) \neq 0$.

Then there exists an open interval $U \subseteq I$ containing $\bar{x}$ and an unique function $h: U \to J$ s.t.

- $h(\bar{x}) = \bar{a}$,
- $F(h(x), x) = x$,
- $h'(\bar{x}) = 0$,
- $h''(\bar{x}) \neq 0$,
- $h(x) \neq \bar{a}$ for $x \in U \setminus \{\bar{x}\}$.

Idea of proof: Similar to Thm 4.6.

$$H(a,x) = \begin{cases} \dfrac{F(a,x) - x}{x - \bar{x}} & x \neq \bar{x}, \\ \lim\limits_{x \to \bar{x}} \dfrac{F(a,x) - x}{x - \bar{x}} & x = \bar{x}. \end{cases}$$

Apply IFT to H (show this is OK).
This proves the existence of $h: U \to J$ s.t. $h(\bar{x}) = \bar{a}$, $H(h(x), x) = 0$.
Show that $F(h(x), x) = x$. Then use implicit derivatives to show that $h''(\bar{x}) \neq 0$.

Notation: Write $F_x = \frac{\partial F}{\partial x}$.

$F(a, x)$.

We know
$$F(h(x), x) = x.$$

Taking deriv. with respect to x:

$$F_a(h(x), x)\, h'(x) + F_x(h(x), x) = 1.$$

Apply it again:

$$F_{aa}(h(x), x)\, h'(x) + F_{ax}(h(x), x) \cdots$$

To save space & time drop arguments

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$$\boxed{(F_{ax} \cdot h' + F_x) h' + F_a \cdot h'' + F_{xa} \cdot h' + F_{xx} = 0}$$

Evaluat this at $x = \bar{x}$. We know $h'(\bar{x}) = 0$.

At $x = \bar{x}$ (i.e. $h(\bar{x}) = \bar{a}$):

$$F_a(\bar{a}, \bar{x})\, h''(\bar{x}) + F_{xx}(\bar{a}, \bar{x}) = 0.$$

$$\boxed{h''(\bar{x}) = \frac{-F_{xx}(\bar{a}, \bar{x})}{F_a(\bar{a}, \bar{x})}.}$$

Note $F_a(\bar{a}, \bar{x}) = 0$ So the red box is undefined......

Do another deriv. and then sub $x = \bar{x}$:

$$\boxed{F_{xxx}(\bar{a}, \bar{x}) + 3 F_{ax}(\bar{a}, \bar{x})\, h''(\bar{x}) = 0} \quad \square$$