Instability via Lyapunov

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Prop 9.13: Suppose x_0 is a fixed point and Ω is an open neighborhood containing x_0. Let $\Omega' \subseteq \Omega$ be open and $G: \Omega \to \mathbb{R}$ a cont. diff. function such that

1. $x_0 \in \overline{\Omega'}$,
2. $\forall x \in \Omega'$, $G(x) > 0$ and $G_f'(x) > 0$,
3. $G(x) = 0$ $\forall x \in \partial\Omega' \cap \Omega$.

Then x_0 is an unstable fixed pt.

proof. Choose $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(x_0)} = \{x \in \mathbb{R}^d : \|x - x_0\| \leq \varepsilon\} \subseteq \Omega.$$

Since $x_0 \in \overline{\Omega'}$, we are finished if each orbit in Ω' leaves $\overline{B_\varepsilon(x_0)}$. Suppose $x(t)$ is a solution where $x(0) \in \Omega'$.

Since $G(x(0)) > 0$, $\exists \delta > 0$ s.t. $G(x(0)) > \delta$.

Then define:

$$\Delta = \{x \in \Omega' \cap \overline{B_\varepsilon(x_0)} : G(x) \geq \delta\}.$$

The set Δ is closed and bounded, so the number

$$k = \min_{x \in \Delta} \{G_f'(x)\} > 0$$

$x \in \Delta$

exists.

Suppose $x(t) \in \Delta$ for all $t \geq 0$. Since Δ is closed and bounded, Δ is compact. Therefore,

$$x_* = \lim_{t \to \infty} x(t) \in \Delta.$$

But this means that

$$0 < k \leq G_f'(x(t))$$

for all $t \geq 0$, which implies

$$G(x_*) = G\left(\lim_{t \to \infty} x(t)\right) = \lim_{t \to \infty} G(x(t)) = \infty.$$

Since Δ is closed and bounded, this a contradiction. Thus, $\exists t > 0$ s.t. $x(t) \notin \Delta$. Since

$$G(x(t)) > G(x(0)) \geq \delta,$$

then, $x(t) \notin B_\varepsilon(x_0)$. $\boxtimes$