Basics of Lyapunov functions

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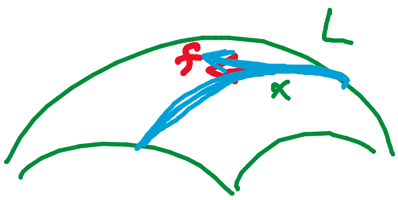
<u>Def:</u> Let $f: \mathbb{R}^d \to \mathbb{R}^d$ and $L: \mathbb{R}^d \to \mathbb{R}$ where f is Lipschitz cont. and L is cont. diff. The orbita derivative of L in the direction of f is the function $L_f': \mathbb{R}^d \to \mathbb{R}$ given by

$$L_f'(x) = \sum_{k=1}^{d} \frac{\partial L}{\partial x_k}(x) \cdot f_k(x)$$

$$= \operatorname{grad}(L)(x) \cdot f(x)$$

$L: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ $f: \mathbb{R}^2 \to \mathbb{R}^2$



L_f' will tells us the slope of L in the direction of f at x.

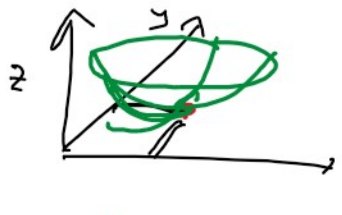
<u>Def:</u> Let x_0 be a fixed pt of f and let $\Omega \subseteq \mathbb{R}^d$ be open with $x_0 \in \Omega$. A cont. diff $L: \Omega \to \mathbb{R}$ is Lyapunov function for $x' = f(x)$ if

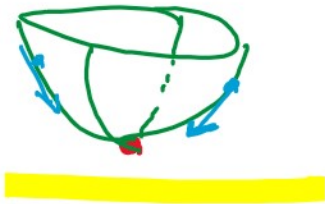
1. $L(x_0) = 0$,
2. $\forall x \in \Omega \setminus \{x_0\}$, $L(x) > 0$,

3. $\forall x \in \Omega \setminus \{x_0\}$, $L_f'(x) \le 0$.

We call L strict Lyapunov function if (3) is strict inequality.

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad L : \mathbb{R}^2 \to \mathbb{R}$





Strong behaviorial implications just for existing.

Downside: No general way to construct these.

Often with relatively examples one can constructions of the form

$$L(x,y) = A(x-x_0)^2 + B(y-y_0)^2.$$

$L(x_0, y_0) = 0$.

For all $(x,y) \in \mathbb{R}^2$, $L(x,y) > 0$.

Ex 9.7:

$$x' = -2x - y$$

$$y' = -x^2 - 4x - y.$$

So $f(x,y) = (-2x-y,\ -x^2-4x-y)$.

A fixed satisfies $f(x_0, y_0) = (0,0)$.

$$y = -2x, \qquad y = -x^2 - 4x$$

$$0 = x^2 + 2x = x(x+2)$$

Two fixed points: $(0,0)$, $(-2,4)$.

$$L(x,y) = (x+2)^2 + (y-4)^2.$$

Satisfies first two properties.

$$L_f'(x,y) = \big(2(x+2),\ 2(y-4)\big) \cdot \big(-2x-y,\ -x^2-4x-y\big)$$

$$= 2(x+2)(-2x-y) + 2(y-4)(-x^2-4x-y)$$

$$= -2x^2y + 4x^2 - 10xy + 24x - 2y^2 + 4y.$$

(Recall, need $L_f' \le 0$)

Taking $L_f'(x,y)$ grad.

$x = -2$
$y = 4$

$$\operatorname{grad} L_f' = \big(-4x(y-2) - 10y + 24,\ -2(x^2 + 5x + 2y - 2)\big)$$

local min. at $(-2,4)$.

Thus, there exists some open neighbor-

Thus, there exists some open neighbor-hood about $(-2, 4)$ s.t. L_f' is nonpos.

So L is a Lyapunov function of f. $\blacksquare$