Characteristic polynomials: the setup

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e^A via characteristic poly of A.

If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$,

$$p_A(\lambda) = \det(\lambda I_d - A).$$

$\exists c_i$ s.t.

$$p_A(\lambda) = \lambda^d + c_{d-1}\lambda^{d-1} + \dots + c_1\lambda + c_0.$$

The characteristic equation:

$$p_A(\lambda) = \lambda^d + c_{d-1}\lambda^{d-1} + \dots + c_1\lambda + c_0 = 0.$$

Ex. 8.10:

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$p_A(\lambda) = \det(\lambda I_3 - A)$$

$A - \lambda I$

$$= \begin{vmatrix} \lambda-1 & -1 & -4 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)^3$$

Note: $p_A(A) = (A - 1\cdot I_3)^3 = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ \dots \end{pmatrix}^3$

Note: $p_A(A) = (A - 1\cdot I_3)^{3} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{nilpotent.}}$

$= 0.$

Thm (Cayley-Hamilton). If $A \in \mathrm{Mat}_{d\times d}(\mathbb{C})$, then $p_A(A) = 0$.

Def: An **entire function** $f: \mathbb{C} \to \mathbb{C}$ is a function that can be expressed as a power series with an infinite radius of convergence. In other words, $\exists$ a complex sequence $(a_n)_{n=0}^{\infty}$ s.t. for all $z \in \mathbb{C}$

$$f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k.$$

An example is $\exp(z)$ or e^z

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

Prop 8.13: Let f be an entire function and p a poly. of degree n. Then $\exists$ an entire function g and a poly q of degree $\leq n-1$ such that

$$f(z) = p(z)g(z) + q(z)$$

$\leq n-1$ such that

$$f(z) = g(z)p(z) + q(z).$$

f: exp

p: char. poly.

$$e^z = g(z)p_A(z) + q(z)$$

$$e^A = g(A)\underbrace{p_A(A)}_{0} + q(A).$$

$$= q(A)$$

Proof: Via induction on n. First n = 1.

$$p(z) = z - c.$$

First, c = 0.

all terms have a multiple of z

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \boxed{\sum_{k=1}^{\infty} a_k z^k} + a_0$$

$$= \left(\sum_{k=1}^{\infty} a_k z^{k-1}\right) z + a_0$$

$$= \underbrace{\left(\sum_{k=0}^{\infty} a_{k+1} z^k\right)}_{g(z)} \cdot \underbrace{z}_{p(z)} + \underbrace{a_0}_{q(z)}.$$

Thus, for n = 1 and c = 0, the Prop. holds.

Now if $c \neq 0$, want

$$f(z) = g(z)(z-c) + q(z)$$

Set $w = z - c$

Set $w = z - c$

$$f(w+c) = g(w+c)(w) + q(w+c)$$

Apply above arguement to the eqn. in w. This proves the base case.

For the induction step assume the Prop. holds for degree $n-1$, so we will show that it holds for degreen n.

Let $p_1(z)$ be a poly of degree $n-1$ s.t.

$$(***) \qquad \underbrace{p(z)}_{n} = \underbrace{p_1(z)}_{n-1} \cdot \underbrace{(z-c)}_{1}.$$

By induction, $\exists g_1$ and q_1:

$$(*) \qquad f(z) = g_1(z)\, p_1(z) + q_1(z),$$

where the degree of q_1 is $n-2$. Since g_1 is an entire function, we can write

$$(**) \qquad \underbrace{g_1}_{\text{entire}} = g \cdot \underbrace{(z-c)}_{\text{"}p\text{"}} + \underbrace{q_0}_{},$$

where q_0 is a constant. Therefore:

$$f(z) \overset{(*)}{=} g_1(z) \cdot p_1(z) + q_1(z)$$

$f(z) = g_1(z) \cdot p_1(z) + q_1(z)$

$$\overset{(**)}{=} \left(g(z)(z-c) + q_0\right) \cdot p_1(z) + q_1(z)$$

$$= g\,\underline{(z-c)p_1} + q_0 p_1 + q_1$$

$$\overset{(***)}{=} \underbrace{g}_{\text{entire}} \cdot \underbrace{p}_{\text{given}} + \underbrace{q_0 p_1 + q_1}_{\text{same poly.}}$$

Set $q(z) = \underbrace{q_0(z) \cdot p_1(z)}_{\leq n-1} + \underbrace{q_1(z)}_{\leq n-2}$, so q has deg. at most $n-1$. $\blacksquare$

Prop 8.14: Let $A \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ with distinct eigenvalues $\lambda_1, \dots, \lambda_m$ with multiplicities $n_1, \dots, n_m$. If f is an entire function, then $\exists$ unique poly. q of degree $d-1$ defined by the conditions $\forall k \in \{1, \dots, m\}$ and $\forall j \in \{0, \dots, n_k - 1\}$,

$$\frac{d^j q}{dz^j}(\lambda_k) = \frac{d^j f}{dz^j}(\lambda_k)$$

such that $f(A) = q(A)$.

Proof: By Prop 8.13, $\exists$ a poly q of deg. at most $d-1$ s.t.

$f(z) = g(z)p(z) + q(z)$

at most $d-1$ s.t.

$$f(z) = g(z)\,p_A(z) + q(z).$$

By C-H, $f(A) = q(A)$. From the assumptions on the eigenvalues of A:

$$p_A(z) = \prod_{k=1}^{m} (z - \lambda_k)^{n_k}.$$

Since

$$\boxed{f(z) - q(z)} = \boxed{g(z) \prod_{k=1}^{m} \underline{(z-\lambda_k)}^{n_k}}$$

it follows that f and q have the same derivatives of orders $1, 2, \ldots, \underline{n_k - 1}$ at $z = \lambda_k$. $\square$